Optimal Mortgage Design*

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Abstract

This paper studies optimal mortgage design. A borrower (a household) with limited liability needs financial support from a lender (a big financial institution) to purchase a house. We characterize the optimal allocation in a continuous time setting in which (i) the borrower's income is volatile and its realization is unobservable to the lender, (ii) the lender has a right to costly foreclose the loan and seize the house, (iii) the borrower's intertemporal consumption preferences are represented by a constant discount factor, (iv) the lender discounts cash flows using a stochastic discount factor that depends on the market interest rate. We show that the optimal allocation can be implemented using either an option adjustable rate mortgage or a combination of an interest only mortgage with a home equity line of credit. Under the optimal contracts, mortgage payments and default rates are higher when the market interest rate is high. However, borrowers benefit from low mortgage payments and low default rates when the market interest rate is low. The gains from using the optimal contract relative to simpler mortgages are substantial and the biggest for those who buy pricey houses given their income level or make little or no downpayment. Thus, our analysis provides theoretical evidence that high concentration of the alternative mortgages in the subprime market can be economically efficient.

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1 Introduction

Recent years have seen a rapid growth in originations of more sophisticated alternative mortgage products (AMPs), such as option adjustable rate mortgages (option ARMs) and interest only mortgages. In the United States, from 2003 through 2005, the originations of AMPs grew from less than 10% of residential mortgage originations to about 30%.[1] As of the first half of 2006, 37%[2] of mortgage originations were AMPs. Option adjustable rate mortgages experienced particularly fast growth. They accounted for as little as 0.5% of all mortgages written in 2003, but their share soared to at least 12.3% through the first five months of 2006.[3] As AMPs have complemented other forms of housing loans rather than replaced them, these nontraditional mortgages account for a significant part of the recent increase in household mortgage debt in the United States, from about 60% of GDP in 2003 to above 75% of GDP in 2006.[4]

Unlike traditional fixed rate mortgages (FRMs) and adjustable rate mortgages (ARMs), AMPs let borrowers pay only the interest portion of the debt or even less than that, while the loan balance can grow above the amount borrowed initially. Often, these mortgages carry teaser rates and come with a second mortgage, taking the form of a home equity line of credit (HELOC). Interest rates on such loans can increase as interest rates in the economy move higher, resulting in increased risk of delinquencies and defaults among AMP borrowers. AMPs are frequently marketed to subprime borrowers - lower-income people with poor credit scores - who often cannot afford a traditional mortgage.[5]

As a result of the recent high increase in delinquency and foreclosure rates among subprime borrowers, AMPs have generated great controversy and criticism. Critics contend that AMPs can hurt borrowers with high interest payments in the future and accuse AMP originators of predatory lending to naive borrowers who do not fully understand mortgage terms.[6] On the other hand, proponents claim that AMPs are more efficient than traditional mortgages because they allow both lenders and borrowers to manage their cash flows intelligently. They maintain that AMPs have helped bring the US homeownership rate to record high in part by extending credit to millions of borrowers who previously would have been denied credit, both for mortgages and for other consumer loans.

Surprisingly, despite of the economic significance of AMPs and the extent of the surrounding controversy, there has so far been no attempt to formally address whether these new mortgages improve benefits to borrowers and lenders relative to traditional mortgages. In this paper, we formally approach this issue by addressing a more general normative question. Assuming full rationality, what is the best possible mortgage contract between a home buyer and a financial institution? Instead of considering a particular class of

[1] United States Government Accountability Office (2006).

[2] Inside Mortgage Finance (2006a).

[3] Data from LoanPerformance, an industry tracker unit of First American Real Estate Solutions (FARES).

[4] The mortgage debt data are from Flow of Funds Accounts of the United States, Federal Reserve Board, and the GDP data are from Bureau of Economic Analysis.

[5] See United States Government Accountability Office (2006).

[6] See for example Department of the Treasury, Board of Governors of the Federal Reserve System, Federal Deposit Insurance Corporation, National Credit Union Administration (2006), or United States Government Accountability Office (2006).

mortgages, we derive an optimal mortgage contract as a solution to a general dynamic contracting problem in a setting with as few assumptions as possible about payments between the borrower and the lender and about circumstances under which the home is repossessed. Then we examine whether features of existing mortgage contracts are consistent with the properties of the best possible contract.

Specifically, we consider a continuous-time setting in which a borrower with limited liability needs outside financial support from a risk-neutral lender in order to purchase a house. Home ownership generates for the borrower a public and deterministic utility stream. The borrower's consumption is divided into two categories: "necessary" consumption, which includes grocery food, medicine, transportation and other goods and services essential for the household survival, and "luxury" consumption, which includes everything else. We assume that the borrower is infinitely risk averse with respect to the necessary consumption and risk neutral with respect to the luxury consumption. The minimum level of necessary consumption is given by an exogenous stochastic process. After paying for his necessary consumption the borrower is free to allocate the remaining part of his income between luxury consumption and debt repayment. Therefore the maximum amount of debt repayment the borrower can make at any given period equals all the luxury consumption he can forgone, i.e. his total income less the necessary consumption spending. The distribution of this "excess" income, which the borrower can use to pay back his debt, is publicly known, however its realizations are privately observable by the borrower. There is a liquidation technology that allows termination of the relationship and transfer of the house to the lender. This transfer of ownership leads to inefficiencies due to associated dead-weight costs. This setting allow us to focus on the fundamental feature of the borrowing-lending relationship with collateral, which is how to efficiently provide a borrower with incentives to repay his debt using a threat of a costly liquidation and at the same time to insure him against the fluctuations in his income.

An important assumption of our model is that the borrower and the lender have different discount rates. The borrower's discount rate γ represents his intertemporal consumption preferences and is constant over time. On the other hand, the lender, a big financial institution, discounts cash flows using a stochastic discount rate r_t that depends on the market interest rate. To the best of our knowledge, this is the first paper that allows for a stochastic interest rate in an optimal dynamic security design setting. We further assume that r_t follows a two-state Markov process and is smaller than the borrower's discount rate. We assume that the borrower is more impatient than the lender reflecting that a borrowing-constrained household has a higher intertemporal marginal rate of substitution then a financial institution.

Before purchase of the house, the borrower and the lender sign a contract that will govern their relationship after the purchase is made. The contract specifies transfers between the borrower and the lender, conditional on the history of the borrower's reports and the circumstances under which the lender would foreclose the loan and seize the home. Although the borrower's reports cannot be verified, the threat of losing ownership of the home induces the borrower to pay his debt.

We characterize the optimal allocation using the borrower's continuation utility a_t and the market interest rate r_t as state variables. Under the optimal allocation, the borrower truthfully reports his income. The home is repossessed when the borrower's continuation utility a_t hits the borrower's reservation utility A for the first time. The borrower consumes part of his excess income whenever a_t reaches the upper boundary $a^1(r_t)$. When $a_t \in [A, a^1(r_t)]$, all the excess income of the borrower is transferred to the lender and so he enjoys no luxury consumption in this region. The borrower's continuation utility increases (decreases) when his excess income realization is high (low).

Interestingly, when the interest rate r_t switches from high to low, the borrower's continuation utility jumps up. On the other hand, when the interest rate r_t switches from low to high, the borrower's continuation utility jumps down, which, in certain cases, can trigger immediate liquidation.[7] This is optimal because the stream of borrower's payments is more valuable for the lender when the interest rate is low. As a result, the chances of home repossession are reduced by moving the borrower's continuation utility further away from the default boundary A when the interest rate switches to low. However, the threat of repossession must be real enough in order for the borrower to share his income with the lender. As a result, the optimal allocation increases the chances of repossession when the interest rate is high in order to compensate for the weakened threat of repossession in the low state. This is done by moving the borrower's continuation utility closer to the default boundary A when the interest rate switches to high.

After characterizing the optimal allocation in terms of the continuation utility of the borrower and the lender, we examine whether features of existing mortgage contracts are consistent with the properties of optimal allocation. We show that the optimal allocation can be implemented in three different ways using combinations of existing residential mortgage instruments. First, it can be implemented using an option ARM with a preferential interest rate. Second it can be implemented using an interest only mortgage with HELOC and two way balance adjustment. Third, it can be implemented using an interest only mortgage with HELOC with a preferential rate and one way balance adjustment.

The option ARM mortgage charges a low preferential interest rate on a portion of the balance. On the remaining part of the balance, a variable rate is charged which positively correlates with the market interest rate. The balance subject to the preferential rate increases when the interest rate switches from high to low and decreases when the interest rate switches from low to high. Also, in general, the borrower's enjoys higher negative amortization limit when the market rate is low and vice versa. The borrower can further indebt himself to finance the interest rate payments or his consumption as long as his debt balance is below the negative amortization limit. The borrower is in default if he is unable to make mortgage payments without exceeding the negative amortization limit. In this case, the lender forecloses the loan and seizes ownership of the home. It is optimal for the borrower to use his income in excess of financing his necessary consumption to make the current interest rate payments and to repay his debt balance. When the borrower realization of the excess income is low the borrower increases his debt balance to finance interest payments, as long as

[7] These relationships are reversed when the likelihood of liquidation is small (when a_t is in the neighborhood of the consumption boundary $a^1(r)$).

he does not exceed the negative amortization limit. Once the borrower repays sufficient amount of debt, so that all his remaining balance is subject to a low preferential interest rate, he spends part of his excess income on luxury consumption.

Under the interest only mortgage with HELOC and two way balance adjustment, the borrower owns a home, while being obligated to make interest coupon payments on the interest only mortgage and interest payments on the home equity credit line balance. The borrower can use the HELOC to finance the interest rate payments or his consumption as long as the HELOC balance is below the credit line limit. The borrower is in default if he is unable to make mortgage payments without exceeding the HELOC credit limit. In this case, the lender forecloses the loan and seizes ownership of the home. The parameters of HELOC are reset every time the market interest rate changes. When the market interest rate switches from high to low, the balance on HELOC is automatically reduced by an amount proportional to the outstanding balance and the interest rate charged on HELOC balance is also reduced. On the contrary, the balance and the HELOC interest rate are automatically increased when the market interest rate switches from low to high. It is optimal for the borrower to use all income he has after financing his necessary consumption to make the current interest rate payments on the interest only mortgage and to repay his HELOC balance. When the realization of the borrower's excess income is low, the borrower draws on the credit line to make the current debt payments, as long as he does not exceed the credit limit. He enjoys luxury consumption only after he repays all his HELOC balance.

Although mortgages with HELOC and two way balance adjustment are interesting from a theoretical point of view, we do not yet observe them in practice. While we actually observe reductions of mortgage debt balance in the form of "cramdown"[8] provisions, the unusual feature of these mortgages is their automatic increase in debt balance in response to a market interest rate increase. The implementation using the interest only mortgage with HELOC with a preferential rate and one way balance adjustment addresses this issue.

The interest only mortgage with HELOC with a preferential rate and one way balance adjustment is similar to the interest only mortgage with HELOC and two way balance adjustment, except that a part of the HELOC balance is subject to a low preferential (teaser) rate and balance adjustment occurs only when the interest rate changes from high to low. This reduction in debt can be interpreted as an automatic "cramdown" provision to be applicable whenever the market interest rate switches to low. When the interest rate changes from low to high, the total amount of the HELOC debt does not change. Instead, the balance subject to the preferential rate shrinks.

All three optimal mortgage implementations provide financial flexibility for the borrower to cover possible low excess income realizations. Given the interest only mortgage with HELOC and two way (or one way) balance adjustment, the borrower can draw on HELOC up to its limit, whenever his excess income is not sufficient to make the coupon payment. Under the option ARM, there is no minimum payment requirement –

[8] "Cramdown" is a court-ordered reduction of the secured balance due on a home mortgage loan, granted to a homeowner who has filed for personal bankruptcy protection.

a low payment from the borrower translates into a higher balance, as long as the balance does not exceed the negative amortization limit. Although home repossession is costly, the borrower does not need to maintain precautionary savings, because the credit commitments by the lender provide a safety net. The borrower only defaults after receiving a sufficient amount of positive shocks to his necessary spending or negative shocks to his total income.[9]

None of the optimal contracts allows borrowers to refinance their mortgages with another lender. Offering this option would limit the ability to provide incentives to the borrower to repay his debt, resulting in a decrease of efficiency of the contract. Therefore, our results lend support to prepayment penalties on refinancing.[10]

This paper shows that the properties of AMPs are consistent with the properties of the optimal mortgage contract, which represents a Pareto improvement over traditional mortgages. Thus, our analysis provides a theoretical evidence that AMPs can benefit both lenders and borrowers, and the explosion in option-ARMs and other exotic mortgages might be driven by their superior efficiency over traditional mortgages.

Critics of AMPs have raised concerns that teaser rates and low minimum payments can result in substantially higher mortgage payments and, as a consequence, higher default rates when interest rates in the economy increase. Nevertheless, this paper demonstrates that this possibility does not necessarily contradict optimality of AMPs. Under the optimal mortgage contract, mortgage payments and default rates are indeed higher when the market interest rate is high. However, borrowers benefit from low mortgage payments and low default rates when the interest rate is low.

The parametrized examples we consider indicate substantial efficiency gains from using the optimal mortgage contract compared to more traditional mortgages. Because AMPs manage default timing more intelligently than simpler mortgages, the gains are the biggest for those who buy pricey houses given their income level or make little or no downpayment. Thus, our results provide a theoretical evidence that high concentration of AMPs in the subprime market may be economically efficient.

Because of high delinquency and foreclosure rates among subprime borrowers, the new federal guidelines issued recently by the Treasury Department, Federal Reserve Board, Federal Deposit Insurance Corporation, and the National Credit Union Administration, are designed to tighten up lenders' underwriting standards for payment-option adjustable rate mortgages, interest-only mortgages, and simultaneous use of home equity line of credits. Also, congress is contemplating a serious tightening of regulations to make the new forms of lending more difficult.[11] We believe that while taking action on predatory lenders could be beneficial, restricting access of the borrowers to these new mortgages might be a bad idea from an economic point

[9] This result is consistent with empirical evidence pointing that consumer delinquency problems are mainly the result of unexpected negative events, that neither the lender nor the borrower could have anticipated at the time the credit request was evaluated (see for example Getter (2003)). In the words of Amy Crews Cutts, deputy chief economist for Freddie Mac, cited by NYT (2007): "If you come in at the edge of affordability, and gas prices go up $100 a month, and insurance premiums go up, and then a water heater breaks, that's the kind of thing that can put a family over the edge."

[10] According to the Wall Street Journal (2005) between 40% and 70% of option ARMs now carry prepayment penalties.

[11] See Joint Economic Committee (2007).

of view. Low default rates of the past might have been economically suboptimal, because many potential homebuyers were shut out of the housing market due to excessively tight underwriting standards.

Related Literature

This paper belongs to the growing literature on dynamic optimal security design, which is a part of the literature on dynamic optimal contracting models using recursive techniques that began with Green (1987), Spear and Srivastava (1987), Abreu, Pearce and Stacchetti (1990), Phelan and Townsend (1991), among many others. Sannikov (2006a) developed continuous-time techniques for a principal-agent problem. The two studies that are most closely related to ours are DeMarzo and Fishman (2004) and its continuous-time formulation by DeMarzo and Sannikov (2006). These papers study long-term financial contracting in a setting with privately observed cash flows, and show that the implementation of the optimal contract involves a credit line with a constant interest rate and credit limit, long-term debt, and equity. Biais et al. (2006) study the optimal contract in a stationary version of DeMarzo and Fishman's (2004) model and show that its continuous time limit exactly matches DeMarzo and Sannikov's (2006) continuous-time characterization of the optimal contract. Tchistyi (2006) considers a setting with correlated cash flows and shows that the optimal contract can be implemented using a credit line with performance pricing. Sannikov (2006b) shows that an adverse selection problem, due to the borrower's private knowledge concerning quality of a project to be financed, implies that, in the implementation of the optimal contract, a credit line has a growing credit limit. Clementi and Hopenhayn (2006) and DeMarzo and Fishman (2006) offer theoretical analyses of optimal investment and security design in moral hazard environments.

Unlike this paper, none of the above studies considers an environment with a stochastic discount rate. We solve for the optimal allocation in the stochastic discount rate environment and find that its implementation involves a variable interest rate charged on the borrower's debt as well as balance adjustments, adjustable preferential debt or a combination of both. On the technical side, building on the martingale techniques developed for Lévy processes, we extend DeMarzo and Sannikov (2006) characterization of the optimal allocation in a continuous-time setting to a stochastic discount rate environment.

There is a sizeable real estate finance literature that addresses the design of mortgages in the presence of asymmetric information between the borrower and lender. The bulk of this literature focuses on adverse selection and how it affects the menu of mortgages being offered to borrowers with limited insurance possibilities. Chari and Jagannathan (1989) consider a model with two private types of borrowers, who differ in terms of the riskiness of their potential gains from selling the property, and show that the optimal contract to be chosen by borrowers with larger potential gains involves contractual arrangements such as points[12] and prepayment penalties together with a "due-on-sale" clause. Brueckner (1994) develops a model in which borrowers self-select into different loans, and shows that the optimal menu of mortgages will induce longer

[12] Points represents the amount paid either to maintain or lower the interest rate charged.

term borrowers to select loans with higher points and a lower coupon. Unlike these two papers, LeRoy (1996) considers a stochastic interest rate environment and finds that, when borrowers refinance optimally, if interest rates fall, the points/coupon choice can at best serve only to separate the least mobile borrower type from all others. Stanton and Wallace (1998) show that in the presence of transaction costs payable by borrowers on refinancing, it is possible to construct a separating equilibrium in which borrowers with differing mobility select fixed rate mortgages with different combinations of coupon rate and points. Posey and Yavas (2001) study how borrowers with different private levels of default risk would self-select between fixed rate mortgages and adjustable rate mortgages, and show the unique equilibrium may be a separating equilibrium in which the high-risk borrowers choose the adjustable rate mortgages, while low-risk borrowers select the fixed rate mortgages. Unlike these papers that focus on adverse selection, Dunn and Spatt (1985) consider a two-period moral hazard model, where future income realization of borrowers are uncertain and private, and show that the optimal mortgage would involve a due-on-sale clause. In terms of this literature, to our knowledge, our paper is the first study of optimal mortgage design in a dynamic moral hazard environment, and the first study that addresses the optimality of alternative mortgage products.

There is also a large literature that focuses on the choice of mortgage contracts and the risk associated with them (for example, Campbell and Cocco (2003)). Unlike our paper, this literature takes a space of contracts as exogenously given, and studies the household choice within this restricted set of contracts. Another branch of research investigates limited participation models, where housing collateral insulates households from labor income shocks. Lustig and Van Nieuwerburgh (2006) typifies this approach.

The paper is organized as follows. Section 2 presents the continuous-time setting of the model. Section 3 introduces the dynamic contracting model with a stochastic discount rate. Section 4 derives the optimal contract. Section 5 presents the implementations of the optimal contract. Section 6 discusses the approximate implementations of the optimal contract. Section 7 studies the efficiency gains due to optimal and approximately optimal contracts. Section 8 concludes.

2 Set-up

Time is continuous and infinite. There is one borrower and one lender (or a group of lenders). The lender (a big financial institution) is risk neutral, has unlimited capital, and values a stochastic cumulative cash flow $\{f_t\}$ as

$$E\left[\int_0^\infty e^{-R_t} df_t\right],$$

where R_t is the market interest rate at which the lender discounts cash flows that arrive at time t. We assume that

$$R_t = \int_0^t r_s ds,$$

where r is an instantaneous interest rate process, which takes values in the set $\{r_L, r_H\}$, where $0 < r_L < r_H$. We assume that r is a continuous-time process adapted to N, where $N = \{N_t, \mathcal{F}_{1,t}; 0 \leq t < \infty\}$ is a standard compound Poisson process with the intensity $\delta(N_t)$ on a probability space $(\Omega_1, \mathcal{F}_1, m_1)$, such that for $t \geq 0$:

$$r_t(N_t) = \begin{cases} r_0 & \text{if } N_t \text{ is even} \\ r_0^c & \text{if } N_t \text{ is odd} \end{cases},$$

$$\delta(N_t) = \begin{cases} \delta(r_0) & \text{if } N_t \text{ is even} \\ \delta(r_0^c) & \text{if } N_t \text{ is odd} \end{cases},$$

where $r_0 \in \{r_L, r_H\}$ is given, and $r_0^c = \{r_L, r_H\} \setminus \{r_0\}$. The above formulation implies that the interest rate process is a first-order time-invariant continuous Markov chain with an exponential distribution with the rate parameter $\delta(r_t)$ of waiting times between successive changes. That is, for any $t \geq 0$,

$$\begin{aligned} P\left[r_{t+s} = r_L \text{ for all } s \in [t, t+\Delta) \,|r_t = r_L\right] &= e^{-\delta(r_L)\Delta}, \\ P\left[r_{t+s} = r_H \text{ for all } s \in [t, t+\Delta) \,|r_t = r_H\right] &= e^{-\delta(r_H)\Delta}. \end{aligned}$$

The borrower's consumption is divided into two categories: "necessary" consumption, which includes grocery food, medicine, transportation and other goods and services essential for the household survival, and "luxury" consumption, which includes everything else. The cumulative minimum level of necessary consumption is given by an exogenous stochastic process $\{\eta_t\}$ that incorporates shocks such as medical bills, auto repair costs, fluctuations of food and gasoline prices and so on. We assume that the borrower is infinitely risk averse with respect to the necessary consumption and risk neutral with respect to the luxury consumption. That is, the borrower's instantaneous utility function is given by

$$u\left(dC_t^0, dC_t\right) = \begin{cases} -\infty, & if\ dC_t^0 < d\eta_t \\ dC_t, & if\ dC_t^0 \geq d\eta_t \end{cases},$$

where $\left\{C_t^0\right\}$ and $\{C_t\}$ denote cumulative flows of the necessary and luxury consumption.

The borrower must use his income to first cover the necessary expenses η_t before spending on luxury consumption. Let $\bar{Y}_t \geq 0$ denote the borrower's total income up to time t. We will focus on the borrower's "excess" income $Y_t \equiv \bar{Y}_t - \eta_t$, which represents a better measure of the borrower's ability to pay for a house than the total income. From now on, we will refer to Y_t and C_t simply as the borrower's income and consumption.

The borrower values a stochastic cumulative consumption flow $\{C_t\}$ as

$$E\left[\int_0^\infty e^{-\gamma t} dC_t\right].$$

We assume that the borrower is impatient, i.e., for all t, $\gamma \geq r_t$. The borrower can buy a home at date $t = 0$ at price P. At any moment in time, ownership of the home generates to the borrower a public and deterministic instantaneous utility equal to θ. The borrower's initial wealth is $Y_0 \geq 0$. We assume that $P > Y_0$, so that the borrower must obtain funds from the lender to finance the purchase of a home.

A standard Brownian motion $Z = \{Z_t, \mathcal{F}_{2,t}; 0 \leq t < \infty\}$ on $(\Omega_2, \mathcal{F}_2, m_2)$ drives the borrower's income process, where $\{\mathcal{F}_{2,t}; 0 \leq t < \infty\}$ is an augmented filtration generated by the Brownian motion. The borrower's income up to time t, denoted by Y_t, evolves according to

$$dY_t = \mu dt + \sigma dZ_t, \tag{1}$$

where μ is the drift of the borrower's disposable income and σ is the sensitivity of the borrower's income to its Brownian motion component. The borrower's income process, Y, is privately observed by him and is meant to represent an excess disposable income he gets after paying all the expenses that have priority[13] over paying off his debt.

In addition, the borrower maintains a private savings account. The private savings account balance, S, grows at the interest rate ρ_t, which is adapted to the process r, and is such that for all t, $\rho_t \leq r_t$. The borrower must maintain a non-negative balance at his account.

At any time, the relationship between the borrower and the lender can be terminated. In this case, the lender receives L, while the borrower receives his reservation value equal to A. We assume that $A \geq \frac{\theta}{\gamma}$ and that

$$r_H L + \gamma A < \theta + \mu,$$

which ensures that the termination of the ongoing relationship is inefficient.

Let $(\Omega, \mathcal{F}, m) := (\Omega_1 \times \Omega_2, \mathcal{F}_1 \times \mathcal{F}_2, m_1 \times m_2)$ be the product space of $(\Omega_1, \mathcal{F}_1, m_1)$ and $(\Omega_2, \mathcal{F}_2, m_2)$.

3 Dynamic Moral Hazard Problem

At time 0, the funds needed to purchase the home in the amount of $P - Y_0$ are transferred from the lender to the borrower. An allocation, (τ, I), specifies a termination time of the relationship, τ, and the transfers between the lender and the borrower that are based on the borrower's report of his income and the realized

[13] This expenditures could include among others: food, health costs, gasoline costs, car maintenance, necessary repairs to maintain house, etc.

interest rate process. Let $\hat{Y} = \left\{\hat{Y}_t : t \geq 0\right\}$ be the borrower's report of his income, where $\hat{Y}$ is (Y, r)-measurable. At any time $0 \leq t \leq \tau$, the allocation transfers the reported amount, $\hat{Y}_t$, from the borrower to the lender, and $I_t(\hat{Y}, r)$ from the lender to the borrower. Below we formally define an allocation.

Definition 1 *An allocation, $\xi = (\tau, I)$, specifies a termination time, τ, and transfers from the lender to the borrower, $I = \{I_t : 0 \leq t \leq \tau\}$, that are based on $\hat{Y}$ and r. Formally, τ is a $(\hat{Y}, r)$-measurable stopping time, and I is a $(\hat{Y}, r)$-measurable continuous-time process, which is such that the process*

$$E\left[\int_0^\tau e^{-\gamma s} dI_s \,|\mathcal{F}_t\right]$$

is square-integrable for $0 \leq t \leq \tau$ and $\hat{Y} = Y$.

The borrower can misreport his income. Consequently, under the allocation $\xi = (\tau, I)$, up to time $t \leq \tau$, the borrower receives a total flow of income equal to

$$\underbrace{(dY_t - d\hat{Y}_t)}_{misreporting} + dI_t,$$

and his private savings account balance, S, grows according to

$$dS_t = \rho_t S_t dt + (dY_t - d\hat{Y}_t) + dI_t - dC_t, \tag{2}$$

where dC_t is the borrower's consumption at time t, which must be non-negative. We remember that, for all $t \geq 0$, $S_t \geq 0$ and $\rho_t \leq r_t$.

In response to an allocation (τ, I), the borrower chooses a feasible strategy that consists of his consumption choice and the report of his income in order to maximize his expected utility. Below we formally define the feasible strategy of the borrower.

Definition 2 *Given an allocation $\zeta = (\tau, I)$, a feasible strategy for the borrower is a pair $(C, \hat{Y})$ such that*

(i) $\hat{Y}$ is a continuous-time process adapted to (Y, r),

(ii) C is a nondecreasing continuous-time process adapted to (Y, r),

(iii) the savings process defined by (2) stays non-negative.

The borrower's strategy is incentive compatible if it maximizes his lifetime expected utility in the class of all feasible strategies given an allocation $\zeta = (\tau, I)$. As a result, we have the following definition.

Definition 3 *Given an allocation $\zeta = (\tau, I)$, the borrower's strategy $(C, \hat{Y})$ is incentive compatible if* *(i) given an allocation $\zeta = (\tau, I)$, the borrower's strategy $(C, \hat{Y})$ is feasible,*

(ii) given an allocation $\zeta = (\tau, I)$, the borrower's strategy $(C, \hat{Y})$ provides him with the highest expected utility among all feasible strategies, that is

$$E\left[\int_0^\tau e^{-\gamma t}(dC_t + \theta dt) + e^{-\gamma\tau} A \,|\mathcal{F}_0\right] \geq E_0\left[\int_0^\tau e^{-\gamma t}(dC'_t + \theta dt) + e^{-\gamma\tau} A \,|\mathcal{F}_0\right]$$

for all the borrower's feasible strategies $(C', \hat{Y}')$, given an allocation $\zeta = (\tau, I)$.

The above definition does not explicitly include the participation constraint imposing the condition that the borrower's utility from the continuation of the allocation should be at least as large as the borrower's outside option, A, which he can receive at any time by quitting. As the borrower can always under-report and steal at rate γA until a termination time, any incentive compatible strategy would yield the borrower utility of at least A.

The above definition of an incentive compatible strategy allows us to define the incentive compatible allocation as follows.

Definition 4 *An incentive compatible allocation is an allocation $\zeta = (\tau, I)$, together with the recommendation to the borrower, $(C, \hat{Y})$, where $(C, \hat{Y})$ is a borrower's incentive compatible strategy given an allocation $\zeta = (\tau, I)$.*

The allocation is optimal if it provides the borrower with his initial expected utility a_0 and maximizes the expected profit of the lender in the class of all allocations that are incentive-compatible. Below we provide a formal definition of the optimal allocation.

Definition 5 *Given the continuation utility to the borrower, a_0, an allocation $\zeta = (\tau^*, I^*)$, together with a recommendation to the borrower $(C^*, \hat{Y}^*)$ is optimal if it maximizes the lender's expected utility:*

$$E\left[\int_0^\tau e^{-R_t}(d\hat{Y}_t - dI_t) + e^{-R_\tau \tau} L \,|\mathcal{F}_0\right]$$

in the class of all incentive-compatible allocations that satisfy the following promise keeping constraint:

$$a_0 = E\left[\int_0^\tau e^{-\gamma t}(dC_t + \theta dt) + e^{-\gamma\tau} A \,|\mathcal{F}_0\right].$$

We note that maximizing the lender's expected utility is equivalent to maximizing the lender's profit, which equals the lender's expected utility less the loan amount to the borrower $[P - Y_0]$, which we take as given.

In the following lemma, we show that searching for optimal allocations, we can restrict our attention to allocations in which truth telling and zero savings are incentive compatible.

Lemma 1 *There exists an optimal allocation in which the borrower chooses to tell the truth and maintains zero savings.*

Proof In the Appendix. ■

The intuition for this result is straightforward. The first part of the result is due to the direct-revelation principle. The second part follows from the fact that it is weakly inefficient for the borrower to save on his private account ($\rho_t \leq r_t$) as any such allocation can be improved by having the lender save and make direct transfers to the borrower. Therefore, we can look for an optimal allocation in which truth telling and zero savings are incentive compatible.

4 Derivation of the Optimal Allocation

In this subsection, we formulate recursively the dynamic moral hazard problem and determine the optimal allocation. First, we consider a problem in which the borrower is not allowed to save and we determine the optimal allocation[14] in this environment. We know from Lemma 1 that it is sufficient to look for optimal allocations in which the borrower reports truthfully and maintains zero savings, and so the optimal allocation of the problem with no private savings, for a given continuation utility to the borrower, yields to the lender at least as much utility as the optimal allocation of the problem when the borrower is allowed to privately save. Finally, we show that the optimal allocation of the problem with no private savings is fully incentive compatible, even when the borrower can maintain undisclosed savings.

Methodologically, our approach is based on continuous-time techniques used by DeMarzo and Sannikov (2006). We extend their techniques to a setting with Lévy processes.

4.1 The Optimal Allocation without Hidden Savings

Consider for a moment the dynamic moral hazard problem in which the borrower is not allowed to save. First, we will find a convenient state space for the recursive representation of this problem. For this purpose, we define the borrower's total expected utility received under the allocation $\xi = (\tau, I)$ conditional on his information at time t, from transfers and termination utility, if he tells the truth:

$$V_t = E\left[\int_0^\tau e^{-\gamma s}\left[dI_s + \theta ds\right] + e^{-\gamma\tau} A \,|\mathcal{F}_t\right].$$

[14] That is the allocation satisfying the properties of Definition 5 and the additional constraint that $S = 0$.

Lemma 2 *The process $V = \{V_t, \mathcal{F}_t; 0 \leq t < \tau\}$ is a square-integrable $\mathcal{F}_t$-martingale.*

Proof follows directly from the definition of process V and the fact that this process is square-integrable, which is implied by Definition 1. ■

Below is a convenient representation of the borrower's total expected utility received under the allocation $\xi = (\tau, I)$ conditional on his information at time t, from transfers and termination utility, if he tells the truth. Let $M = \{M_t = N_t - t\delta(N_t), \mathcal{F}_{1,t}; 0 \leq t < \infty\}$ be a compensated compound Poisson process.

Proposition 1 *There exists $\mathcal{F}_t$-predictable processes $(\beta, \psi) = \{(\beta_t, \psi_t); 0 \leq t \leq \tau\}$ such that*

$$\begin{aligned} V_t &= V_0 + \int_0^t e^{-\gamma s}\beta_s dZ_s + \int_0^t e^{-\gamma s}\psi_s dM_s = \\ &\quad V_0 + \int_0^t e^{-\gamma s}\beta_s \underbrace{\left(\frac{dY_s - \mu ds}{\sigma}\right)}_{dZ_s} + \int_0^t e^{-\gamma s}\psi_s (dN_s - \delta(N_s)ds). \end{aligned} \tag{3}$$

Proof We note that the couple (Z, N) is a Brownian-Poisson process, and it is an independent increment process, which is a Lévy processes, on the space $(\Omega, \mathcal{F}, m)$. Then, Theorem III.4.34 in Jacod and Shiryaev (2003) gives us the above martingale representation for a square-integrable martingale adapted to $\mathcal{F}_t$ taking values in a finite dimensional space (the process V). ■

According to the martingale representation (3), the total expected utility of the borrower under the allocation $\xi = (\tau, I)$ and truth telling conditional on his information at time t equals its unconditional expectation plus two terms that represent the accumulated effect on the total utility of, respectively, the income uncertainty revealed up to time t (Brownian motion part), and the interest rate uncertainty that has been revealed up to time t (compensated compound Poisson part).

According to Proposition 1, when the borrower reports truthfully, his total expected utility under the allocation $\xi = (\tau, I)$ conditional on the termination time τ equals

$$V_\tau = V_0 + \int_0^\tau e^{-\gamma s}\beta_s \left(\frac{dY_s - \mu ds}{\sigma}\right) + \int_0^\tau e^{-\gamma s}\psi_s dM_s.$$

As I and τ depend exclusively on the borrower's report $\hat{Y}$ and the public interest rate process r, when the borrower reports $\hat{Y}$, by (3) he gets the expected utility, $a_0(\hat{Y})$, which equals

$$\begin{aligned} a_0(\hat{Y}) &= E\left[V_0 + \int_0^\tau e^{-\gamma t}\beta_t \left(\frac{d\hat{Y}_t - \mu dt}{\sigma}\right) + \int_0^\tau e^{-\gamma t}\psi_t dM_t + \underbrace{\int_0^\tau e^{-\gamma t}(dY_t - d\hat{Y}_t)}_{\text{utility from stealing}} |\mathcal{F}_0\right] = \\ &E\left[V_0 + \int_0^\tau e^{-\gamma t}\beta_t \left(\frac{dY_t - \mu dt}{\sigma}\right) + \int_0^\tau e^{-\gamma t}\left(1 - \frac{\beta_t}{\sigma}\right)\left(dY_t - d\hat{Y}_t\right) + \int_0^\tau e^{-\gamma t}\psi_t dM_t \,|\mathcal{F}_0\right]. \end{aligned} \tag{4}$$

Note that because the process $(\beta,\psi) = \{(\beta_t,\psi_t); 0 \le t \le \tau\}$ is $\mathcal{F}_t$−predictable, as for any $t \ge 0$, $s \ge 0$, $E_0\left[Z_{t+s} - Z_t \,|\mathcal{F}_0\right] = E_0\left[M_{t+s} - M_t \,|\mathcal{F}_0\right] = 0$, and given that $E\left[V_0 \,|\mathcal{F}_0\right] = V_0$, we have that

$$a_0(\hat{Y}) = V_0 + E\left[\int_0^\tau e^{-\gamma t}\left(1 - \frac{\beta_t}{\sigma}\right)\left(dY_t - d\hat{Y}_t\right)|\mathcal{F}_0\right]. \tag{5}$$

Representation (5) leads us to the following formulation of incentive compatibility.

Proposition 2 *If the borrower cannot save, truth telling is incentive compatible if and only if* $\beta_t \ge \sigma$ $(m - a.s.)$ *for all* $t \le \tau$.

Proof Immediately follows from (5). ■

It is important to stress that in providing incentives for truth telling one can neglect an impact of reporting strategies on the magnitude of the adjustments, ψ, in the borrower's continuation utility that occurs when the lender's interest rate changes. It follows from (4) that, though in principle the reporting strategy of the borrower does affect the magnitude of these adjustments, from the perspective of the borrower such adjustments have zero effect on the borrower's expected utility whatever is his reporting strategy. This property considerably simplifies the formulation of incentive compatibility.

To characterize the optimal allocation recursively, we define the borrower's continuation utility at time t if he tells the truth as

$$a_t = E\left[\int_t^\tau e^{-\gamma(s-t)}\left[dI_s + \theta ds\right] + e^{-\gamma(\tau-t)}A\,|\mathcal{F}_t\right].$$

Note that for $t \le \tau$ we have that

$$V_t = \int_0^t e^{-\gamma s}(dI_s + \theta dt) + e^{-\gamma t}a_t.$$

But this, together with (3), implies the following law of motion of the borrower's continuation utility:

$$da_t = \gamma a_t dt - \theta dt - dI_t + \beta_t dZ_t + \psi_t dM_t = \left(\gamma a_t - \theta - \psi_t \delta(r_t)\right)dt - dI_t + \beta_t dZ_t + \psi_t dN_t. \tag{6}$$

Here we discuss informally, using the dynamic programming approach, how to find out the most efficient way to deliver a borrower any continuation utility $a \ge A$. The proof of Proposition 3 formalizes our discussion below. Let $b(a,r)$ be the highest expected utility of the lender that can be obtained from an incentive compatible allocation that provides the borrower with utility equal to a given that the current interest rate is equal to r. To simplify our discussion we assume that the function b is concave and C^2 in its first argument. Let b' and b'' denote, respectively, the first and the second derivative of b with respect to the borrower's continuation utility a.

We start by observing that transferring lump-sum dI from the lender to the borrower with continuation utility a, moves an allocation to that of the borrower's continuation utility of $a - dI$. The efficiency implies that

$$b(a,r) \geq b(a-dI,r) - dI, \tag{7}$$

which shows that for all $(a,r) \in [A,\infty)\times\{r_L,r_H\}$ the marginal cost of delivering the borrower his continuation utility can never exceed the cost of an immediate transfer in terms of the lender's utility, that is

$$b'(a,r) \geq -1.$$

Define $a^1(r)$, $r \in \{r_L,r_H\}$, as the lowest value of a such that $b'(a,r) = -1$. Then, it is optimal to pay the borrower as follows

$$dI(a,r) = \max(a - a^1(r), 0).$$

These transfers, and the option to terminate, keep the borrower's continuation utility between A and $a^1(r)$. But this implies that when $a \in [A, a^1(r)]$, and when the borrower is telling the truth, his continuation utility evolves according to

$$da_t(r_t) = (\gamma a_t - \theta - \delta(r_t)\psi_t)\,dt + \beta_t dZ_t + \psi_t dN_t. \tag{8}$$

We need to characterize the optimal choice of process (β_t,ψ_t), where $\frac{\beta_t}{\sigma}$ determines the sensitivity of the borrower's continuation utility with respect to his report, and ψ_t determines the adjustment of the borrower's continuation utility due to a change in the interest rate. Using Ito's lemma, we find that

$$\begin{aligned} db(a_t,r_t) &= (\gamma a_t - \theta - \psi_t\delta(r_t))b'(a_t,r_t)dt \\ &\quad + \frac{1}{2}\beta_t^2 b''(a_t,r_t)dt + \beta_t b'(a_t,r_t)dZ_t + \left[b(a_t+\psi_t,r_t^c) - b(a_t,r_t)\right]dN_t, \end{aligned}$$

where $r_t^c = \{r_L,r_H\}\setminus\{r_t\}$. Using the above equation, we find that the lender's expected cash flows and the change in the value he assigns to the allocation are given as follows:

$$E\left[dY_t + db(a_t,r_t)\,|\mathcal{F}_t\right] =$$
$$\left[\mu + (\gamma a_t - \theta - \psi_t\delta(r_t))b'(a_t,r_t) + \frac{1}{2}\beta_t^2 b''(a_t,r_t) + \delta(r_t)\left(b(a_t+\psi_t,r_t^c) - b(a_t,r_t)\right)\right]dt.$$

From Proposition 2, we know that if $\beta_t \geq \sigma$ for all $t \leq \tau$ then the borrower's best response strategy is to report the truth, that is, $\hat{Y} = Y$. Because at the optimum, at any time t, the lender should earn an instantaneous total return equal to the interest rate, r_t, we have the following Bellman equation for the value function of the lender

$$r_t b(a_t,r_t) =$$
$$\max_{\beta_t\geq\sigma,\ \psi_t\geq A-a_t}\left[\mu + (\gamma a_t - \theta - \psi_t\delta(r_t))b'(a_t,r_t) + \frac{1}{2}\beta_t^2 b''(a_t,r_t) + \delta(r_t)\left(b(a_t+\psi_t,r_t^c) - b(a_t,r_t)\right)\right]. \tag{9}$$

Given the concavity of the function $b(\cdot, r_t)$, $b''(a_t, r_t) = \frac{d^2 b(a_t, r_t)}{da_t^2} \leq 0$, setting

$$\beta_t = \sigma$$

for all $t \leq \tau$ is optimal. The concavity of the objective function in ψ_t in the RHS of the Bellman equation (9) also implies that the optimal choice of ψ_t is given as a solution to

$$b'(a_t, r_t) = b'(a_t + \psi_t, r_t^c), \tag{10}$$

provided that $\psi_t > A - a_t$, and otherwise $\psi_t = A - a_t$. Note that this implies that $\psi_t = \psi(a_t, r_t)$.

The lender's value function therefore satisfies the following differential equation

$$r_t b(a_t, r_t) = \mu + (\gamma a_t - \theta - \psi(a_t, r_t)\delta(r_t))b'(a_t, r_t) + \frac{1}{2}\sigma^2 b''(a_t, r_t) + \delta(r_t)\left(b(a_t + \psi(a_t, r_t), r_t^c) - b(a_t, r_t)\right) \tag{11}$$

with $b(a_t, r_t) = b(a^1(r_t), r_t) - (a - a^1(r_t))$ for $a_t > a^1(r_t)$ and the function ψ specified above.

We need some boundary conditions to pin down a solution to this equation and the boundaries $a^1(r)$, $r \in \{r_L, r_H\}$. The first boundary condition arises because the relationship must be terminated to hold the borrower's value to A, so $b(A, r_t) = L$. The second boundary condition comes from the fact that the first derivatives must agree at the boundary, so $b'(a^1(r_t), r_t) = -1$. The final boundary condition is the condition for the optimality of $a^1(r_t)$, which requires that the second derivatives match at the boundary. This condition implies that $b''(a^1(r_t), r_t) = 0$, or equivalently, using equation (11), that

$$r_t b(a^1(r_t), r_t) = \mu + \theta - \gamma a^1(r_t) + \delta(r_t)\left[\psi(a^1(r_t), r_t) + b(a^1(r_t) + \psi(a^1(r_t), r_t), r_t^c) - b(a^1(r_t), r_t)\right]. \tag{12}$$

By definition, $a^1(r)$ is the lowest value of a such that $b'(a, r) = -1$, thus

$$\psi(a^1(r_L), r_L) = -\psi(a^1(r_H), r_H) = a^1(r_H) - a^1(r_L).$$

This, combined with (12) implies that

$$\mu + \theta = r_L b(a^1(r_L), r_L) + \gamma a^1(r_L) - \delta(r_L)\left[b(a^1(r_H), r_H) - b(a^1(r_L), r_L) + a^1(r_H) - a^1(r_L)\right],$$
$$\mu + \theta = r_H b(a^1(r_H), r_H) + \gamma a^1(r_H) - \delta(r_H)\left[b(a^1(r_L), r_L) - b(a^1(r_H), r_H) + a^1(r_L) - a^1(r_H)\right].$$

The proposition below formalizes our findings.

Proposition 3 *Let b be a C^2 function (in a) that solves:*

$$rb(a, r) = \mu + (\gamma a - \theta - \psi(a, r)\delta(r))b'(a_t, r_t) + \frac{1}{2}\sigma^2 b''(a, r) + \delta(r_t)\left(b(a_t + \psi(a, r), r^c) - b(a, r)\right), \tag{13}$$

when a is in the interval $[A, a^1(r)]$, and $b'(a,r) = -1$ when $a > a^1(r)$, with boundary conditions $b(A,r) = L$ and

$$\begin{aligned}
\mu + \theta &= r_L b(a^1(r_L), r_L) + \gamma a^1(r_L) - \delta(r_L)\left[b(a^1(r_H), r_H) - b(a^1(r_L), r_L) + a^1(r_H) - a^1(r_L)\right], \\
\mu + \theta &= r_H b(a^1(r_H), r_H) + \gamma a^1(r_H) - \delta(r_H)\left[b(a^1(r_L), r_L) - b(a^1(r_H), r_H) + a^1(r_L) - a^1(r_H)\right],
\end{aligned}$$

and where

$$\psi(a,r) = \begin{cases} \text{is a } C^1 \text{ (in } a\text{) solution to } b'(a,r) = b'(a+\psi, r^c) \text{ for all } (a,r) \\ \text{for which the solution is such that } \psi(a,r) > A - a \\ \\ \text{otherwise it is equal to } A - a \end{cases}, \tag{14}$$

where $r \in \{r_L, r_H\}$ and $r^c = \{r_L, r_H\} \setminus \{r\}$.

Then the optimal allocation that delivers to the borrower the value a_0 takes the following form:

(i) If $a_0 \in [A, a^1(r_0)]$, $r_0 \in \{r_L, r_H\}$, a_t evolves as

$$da_t(r_t) = (\gamma a_t dt - \theta dt - dI_t) + (d\hat{Y}_t - \mu dt) + \psi(a_t, r_t)(dN_t - \delta(r_t)dt), \tag{15}$$

and

- *when $a_t \in [A, a^1(r_t))$, $dI_t = 0$,*
- *when $a_t = a^1(r_t)$ the transfers dI_t cause a_t to reflect at $a^1(r_t)$.*

(ii) If $a_0 > a^1(r_0)$ an immediate transfer $a_0 - a^1(r_0)$ is made.

The relationship is terminated at time τ when a_t hits A. The lender's expected utility at any time t is given by the function $b(a_t, r_t)$ defined above, which is strictly concave in a_t over $[A, a^1(r_t)]$.

Proof In the Appendix. ■

The evolution of the continuation utility (15) implied by the optimal allocation serves three objectives - promise keeping, incentives, and efficiency. The first component of (15) accounts for promise keeping. In order for a_t to correctly describe the lender's promise to the borrower, it should grow at the borrower's discount rate, γ, less the payment, θdt, he receives from owning the home, and less the flow of payments, dI_t, from the lender.

The second term of (15) provides the borrower with incentives to report truthfully his income to the lender. Because of inefficiencies resulting from liquidation, reducing the risk in the borrower's continuation utility lowers the probability that the borrower's expected utility reaches A, and thus lowers the probability of costly liquidation. Therefore, it is optimal to make the sensitivity of the borrower's continuation utility with respect to its report as small as possible provided that it does not erode his incentives to tell the truth. The minimum volatility of the borrower's continuation utility with respect to his report of income required for truth-telling equals 1. To understand this, note that, under this choice of volatility, underreporting income by one unit would provide the borrower with one additional unit of current utility through increased consumption, but would also reduce the borrower's continuation utility by one unit, so that this volatility provides the borrower with just enough incentives to report a true realization of income. Note that when the borrower reports truthfully, the term $\left(d\hat{Y}_t - \mu dt\right)$ is driftless and equals to σdZ_t.

The last term of (15) captures the effects of changes in the lender's interest rate process on the borrower's continuation utility. The optimal adjustments, ψ, in the borrower's continuation utility, which are applicable when there is a change in the lender's interest rate, are such that the sensitivity of the lender's expected utility, b, with respect to the borrower's continuation utility, a, is equalized just before and after an adjustment is made.[15] This sensitivity represents an instantaneous marginal cost of delivering the borrower his continuation utility in terms of the lender's utility, and so the efficiency calls for equalizing this cost across the states. We note that these adjustments imply the compensating trend in the borrower's continuation utility, $-\delta(r_t)\psi(a_t, r_t)dt$, which exactly offsets the expected effect these adjustments have on the borrower's expected utility.

Below we state a useful lemma that characterizes the behavior of the optimal allocation when the borrower's continuation utility is close to liquidation and there is an interest rate change.

Lemma 3 *At the optimal allocation, there exists* $\bar{a} \in (A, a^1(r_L)]$ *such that*

- $\psi(A, r_H) = \bar{a} - A$,
- $\psi(a, r_L) = A - a$ *for* $a \in [A, \bar{a}]$.

Proof From the definition of function b and the fact that $r_L < r_H$ it follows that, for any $a > A$, $b(a, r_L) > b(a, r_H)$. This, together with $b(A, r_L) = b(A, r_H) = L$, implies that $b'(A_+, r_L) > b'(A_+, r_H)$. Let $\bar{a}$ be the smallest $a > A$ such that $b'(a, r_L) = b'(A_+, r_H)$. The existence of such $\bar{a}$ follows from the fact that, for any $a \in [A, a^1(r_t)]$, $b'(a, r_t) \geq -1$ and $b'(a^1(r_t), r_t) = -1$. This combined with (14) yields us the alleged properties of function ψ. ■

Corollary 1 *Lemma 3 implies that under the optimal allocation, whenever* $a_t \in (A, \bar{a}]$*, an instantaneous increase of the interest rate,* r_t*, triggers the termination of the relationship.*

[15] Provided that the solution to (10) is interior.

4.2 The Optimal Allocation with Hidden Savings

So far we have characterized the optimal allocation under the assumption that the borrower cannot save. Now we show that, given the optimal allocation of the problem with no hidden savings, the borrower has no incentive to save at the solution, and thus the allocation of Proposition 3 is also optimal in the environment where the borrower can privately save.

Proposition 4 *Suppose that the process a_t is bounded above and solves*

$$da_t = \gamma a_t dt - \theta dt - dI_t + (d\hat{Y}_t - \mu dt) + \psi_t dM_t \tag{16}$$

until stopping time $\tau = \min\{t\,|a_t = A\}$, where ψ_t is an $\mathcal{F}_t-$predictable process. Then the borrower's expected utility from any feasible strategy in response to an allocation (τ, I) is at most a_0. Moreover, the borrower attains the expected utility a_0 if the borrower reports truthfully and maintains zero savings.

Proof In the Appendix. ■

The above proposition shows that allocations from a broad class, including the optimal allocation of Proposition 3, remain incentive-compatible even if the borrower is allowed to privately save.

4.3 An Example

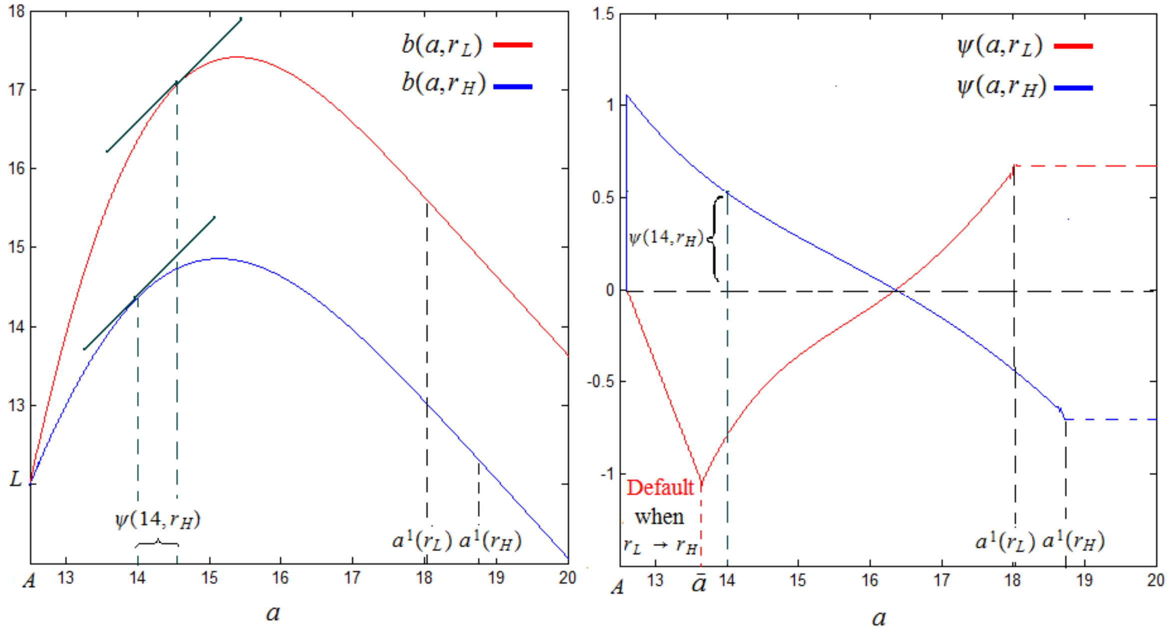
In this section we illustrate the features of the optimal allocation in a parametrized example. Table 1 shows the parameters of the model.

Table 1. Parameters of the model

Interest rate process				Borrower's discount rate	Income process		Utility flow from home	Liquidation values	
r_L	r_H	$\delta(r_L)$	$\delta(r_H)$	γ	μ	σ	θ	A	L
1.5%	6.5%	0.12	0.12	8%	1	1	1	12.5	12

The left hand-side of Figure 1 shows the lender's value function at both interest rates as a function of the borrower's continuation utility. For a given continuation utility to the borrower, the value function of the lender at the low interest rate is always above the one at the high interest rate, except at termination when they are equal, as the lender attaches more value to the proceeds from the continuation of the relationship when his discount rate is lower. As we observe, it is optimal to allow the borrower to consume his disposable income earlier when the interest rate is low, that is $a^1(r_L) < a^1(r_H)$. Intuitively, when the lender's interest rate is low, it is more costly to postpone borrower's consumption, as tension between the borrower's valuation

Figure 1: The lender's value function and the optimal adjustments in the borrower's continuation utility.



of future payoffs and that of the lender is larger. To reduce this cost, it is optimal to allow the borrower to consume his excess disposable income earlier.

The right hand-side of Figure 1 shows the optimal adjustments in the borrower's continuation utility, ψ, applicable when there is a change in the market interest rate. The borrower's continuation utility increases with a decrease in the interest rate and decreases with an interest rate increase, except in the area close to the reflection barriers when this relationship is reversed. The size of these adjustments is proportional to the distance of the borrower's continuation utility from the termination cutoff of A.

The optimal adjustment of the borrower's continuation utility, ψ, is shaped by two competing forces stemming from, respectively, the costly termination of the relationship and the difference in the discount rates. The closer the borrower's continuation utility is to the termination boundary A, the bigger is the role played by the costly termination in shaping the optimal adjustment function. It is efficient to reduce the chances of costly termination when the interest rate falls, as the stream of transfers from the borrower is more valuable for the lender when the interest rate is low. A reduction in the likelihood of termination is engineered by influencing the borrower's continuation utility in two ways. First, it is optimal to instantaneously increase the borrower's continuation utility if the market interest rate falls, and this is even more so the more likely the relationship is to be terminated. Second, it is optimal to introduce a positive trend in the law of motion of the borrower's continuation utility, which reinforces the first adjustment over time to the extent the interest rate stays low. As a result of these adjustments, the chances of costly home repossession are reduced by moving the borrower's continuation utility further away from the termination boundary A. However, the threat of repossession must be real enough in order for the borrower to share his income with the lender. As a result, the optimal allocation increases the chances of repossession when the interest rate is high in order to compensate for the weakened threat of repossession in the low-interest state, both by instantaneously decreasing the borrower's continuation utility and by introducing a negative trend in its law of motion.

If the borrower's continuation utility is distant from the termination boundary A, then, intuitively, the discrepancy in the discount rates begins to play the dominant role in shaping the optimal adjustment function, as the likelihood of termination is small. When the lender's interest rate switches to low, there is more tension between the borrower's valuation of future payoffs and that of the lender, and thus it is more costly to postpone the borrower's consumption, the more so the bigger is his continuation utility. To reduce this cost, it is optimal to decrease the borrower's continuation utility when the interest rate falls, by both an instantaneous adjustment and a negative time trend, provided that his prior continuation utility was sufficiently large. In order to compensate for this reduction in the borrower's continuation utility when the interest rate switches to low, his continuation utility is increased to a range of high values of the borrower's continuation utility when the interest rate increases. It is important to observe that the adjustment of the borrower's continuation utility in this region has second order welfare effects. This is because there is less difference between the slopes of the lender's value function at the low and at the high interest rate state, the further away the borrower's continuation utility is from the termination boundary A. We will use this fact in Section 6, where we simply ignore the adjustments of the borrower's continuation utility in a region close to the reflection barriers.

5 Implementations of the Optimal Allocation

So far, we have characterized the optimal allocation in terms of the transfers between the borrower and the lender and liquidation timing. In this section, we show that the optimal allocation can be implemented using financial arrangements that resemble the ones used in the residential mortgage market.

We consider different ways to implement the optimal allocation. First, we show that the optimal allocation can be implemented using an option adjustable rate mortgage (option ARM) with a preferential interest rate. The practical advantage of this implementation is that the mortgage balance is not directly affected by changes in the interest rates in the economy. Second, we discuss alternative implementations: (i) an interest only mortgage with HELOC and two way balance adjustment and (ii) an interest only mortgage with HELOC with a preferential rate and one way balance adjustment.

We start with the following definition.

Definition 6 *The mortgage contract is optimal if it implements the optimal allocation of Proposition 3.*

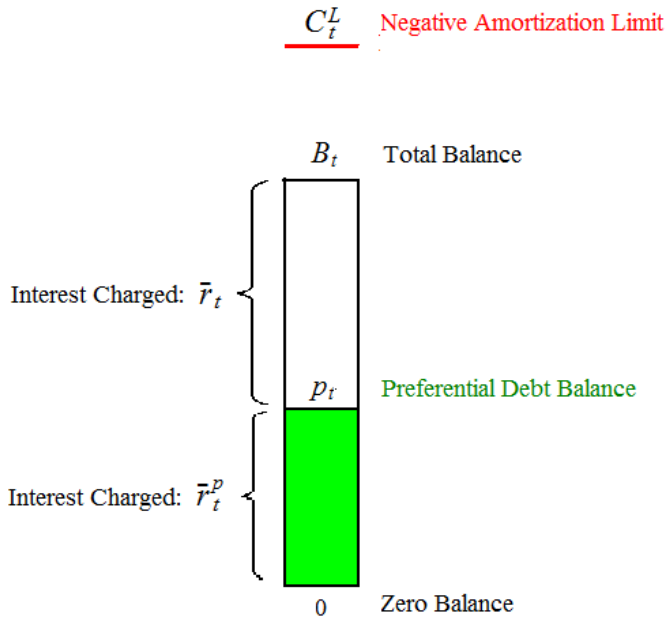


Figure 2: Option ARM structure

5.1 Option ARM

In this section, we consider an option ARM. This is an adjustable rate mortgage with no minimum payment. A part of the mortgage debt is subject to the preferential interest rate. The definition below provides a formal description of this class of mortgage contracts.

Definition 7 *An option adjustable rate mortgage with a preferential interest rate consists of:*

- *Mortgage loan with a time-t negative amortization limit equal to C_t^L. If the balance of the loan exceeds the negative amortization limit, default occurs, in which case the lender repossesses the home.*

- *At any time t, an instantaneous interest rate on the balance, B_t, is equal to a preferential rate $\bar{r}_t^p$ on a part of the balance below p_t, and $\bar{r}_t$ on the part of the balance above p_t.*

Figure 2 graphically demonstrates features of option ARM.

The proposition below shows that the optimal allocation can be implemented with an option ARM with a preferential interest rate.

Proposition 5 *There exists an optimal option adjustable rate mortgage with a preferential interest rate that has the following features*

$$\bar{r}_t(B_t - p_t, r_t) = \gamma + \delta(r_t) \frac{\left[\psi(a^1(r_t) - (B_t - p_t), r_t) - \psi(a^1(r_t), r_t)\right]}{B_t - p_t}, \ if \ B_t \geq p_t \tag{17}$$

$$\bar{r}_t^p(p_t, r_t) = \frac{\theta + \mu - \gamma a^1(r_t) + \delta(r_t)\psi(a^1(r_t), r_t)}{p_t} \tag{18}$$

$$C_t^L(p_t, r_t) = p_t + a^1(r_t) - A \tag{19}$$

$$dp_t = \begin{cases} \left[\psi(a^1(r_t) - (B_t - p_t), r_t) - \left(a^1(r_t^c) - a^1(r_t)\right)\right] dN_t, \ if \ B_t \geq p_t \\ 0, \ if \ B_t < p_t \end{cases} . \tag{20}$$

Under the terms of this mortgage, it is optimal for the borrower to use all available cash flows to pay down balance B_t when $B_t > p_t$, and consumes all excess cash flows once the balance drops to p_t. For balance $B_t \geq p_t$, the borrower's continuation utility a_t is equal to

$$a_t = A + \left[C_t^L(p_t, r_t) - B_t\right] = a^1(r_t) - (B_t - p_t) \tag{21}$$

If the preferential rate reaches its upper boundary γ, the mortgage is reset to a more complicated contract that implements the continuation of the optimal allocation.

Proof In the Appendix. ■

Remark 1 *The initial balance B_0 should be greater than or equal to p_0. If $B_0 < p_0$, it is optimal for the borrower to consume $p_0 - B_0$ immediately by increasing the balance to p_0.*

Remark 2 *The negative amortization limit equal to C_t^L, preferential balance p_t, and the preferential interest rate $\bar{r}_t^p$ are reset only when the interest rate in the economy changes.*

Remark 3 *When default happens, the lender receives the liquidation value L of the home, and the borrower obtains the value A of his outside option.*

Remark 4 *In order for the mortgage to remain incentive compatible, the preferential rate $\bar{r}_t^p$ must stay below γ. Although it is theoretically possible, our simulations show that for most parameters chances of the preferential rate reaching its upper boundary γ over a period of 30 years are extremely small.*

Remark 5 *In the proposed implementation, parameter p_0 at time zero can be chosen arbitrarily, provided interest rate $\bar{r}_0^p$ given by (18) is no greater than γ. One way to initiate the mortgage is to have the market value of the mortgage equal to the book value:*

$$B_0 = b\left(r_0, p_0 + a^1(r_0) - B_0\right).$$

How does the optimal option adjustable rate mortgage implement the optimal allocation? The debt balance above the debt limit subject to the preferential interest rate can be considered as a memory device that summarizes all the relevant information regarding the past cash flow realizations revealed by the borrower through repayments. The interest rates charged on the balance, along with the preferential debt limit, and the negative amortization limit are chosen so that equation (21) always holds. This ensures incentive compatibility of the mortgage. Indeed, at any time t the borrower can consume all his available credit $C_t^L(p_t, r_t) - B_t$ and default immediately. However, (21) implies that the payoff from this strategy is equal to the expected utility a_t the borrower would obtain by postponing consumption until his debt balance is reduced to the preferential debt limit.

The adjustable features of the above mortgage contract are needed to implement the effects of the changes in the interest rate on the borrower's continuation utility. In the optimal option ARM, the adjustments of the debt subject to the preferential rate (20) and the adjustments to the negative amortization limit implement all instantaneous adjustments in the borrower's promised utility that are applicable when the lender's interest rate changes. The variable component of the interest rate (28) guarantees that a change in the borrower's promised utility implied by the mortgage contract includes the trend that compensates the borrower, in expectation, for the instantaneous adjustments in his promise utility that happen when the interest rate changes.

The fixed component of the interest rate (17) on the debt above the preferential debt limit insures that under the optimal strategy of the borrower, given the above mortgage contract, the borrower's promised utility would be increased at the rate of γ as in the optimal allocation. The preferential interest rate insures that an above-average income realization and so an above-average repayment increases the borrower's promised utility, which corresponds here to a decrease in his debt balance, and vice versa.

To further characterize the above mortgage contract we, will restrict our attention to the environment in which the optimal contract satisfies the following condition.[16]

Condition 1 *Function ψ implied by the optimal allocation is such that $\psi(a, r_L)$ is strictly increasing in a for $a \in [\bar{a}, a^1(a_L)]$, and so $\psi(a, r_H)$ is strictly decreasing in a for $a \in [A, a^1(a_H)]$, where $\bar{a}$ is defined as in Lemma 3.*

Parameters C_t^L, p_t, $\bar{r}_t^p$ and $\bar{r}_t$ are reset each time the interest rate in the economy changes. This is needed to take into account the effects that the interest rate in the economy has on the borrower's continuation utility. As the following corollary shows, under the optimal option ARM with preferential interest rate, whenever the debt balance is close to the negative amortization limit, an increase in the interest rate would cause default on the mortgage. If the optimal adjustment function, ψ, satisfies the properties of Condition 1, a decrease in the lender's interest rate results in an increase in the amount of debt subject to the preferential rate and vice versa. In addition, interest rate $\bar{r}_t$ positively correlates with the lender's interest rate.

[16] This condition holds in all parametrized examples we considered.

Corollary 2 *The optimal option ARM with preferential interest rate has the following properties:*

i) Let $\bar{B}_t = p_t + a^1(r_L) - \bar{a}$ where $\bar{a}$ is defined as in Lemma 3. Then, whenever $B_t \in [\bar{B}_t, C_t^L(p_t, r_L))$, an instantaneous increase in the interest rate in the economy triggers mortgage default;

ii) Suppose further that function ψ corresponding to the optimal contract satisfies the properties of Condition 1, then,

- $dp_t < 0$, *whenever interest rate r_t increases,*
- $dp_t > 0$, *whenever interest rate r_t decreases,*
- *for any $B' \in [p_t, C_t^L(r_L)]$ and $B'' \in [p_t, C_t^L(r_H)]$,*

$$\bar{r}_t(B' - p_t, r_L) < \gamma < \bar{r}_t(B'' - p_t, r_H)$$

Proof Corollary 2 follows from Proposition 5 and Lemma 3. ■

Since it is optimal for the borrower to use all available cash flows to pay down balance B_t as long as $B_t > p_t$, lower rates on the mortgage do not necessarily reduce the mortgage payments. It is optimal to reduce interest rates, and as a result default rates, when the interest rate in the economy is low because the stream of borrower's payments is more valuable for the lender when the lender discounts them with lower interest rate. On the other hand, the threat of repossession must be real enough for the borrower to share his income with the lender. As a result the optimal option ARM increases the chances of repossession by charging higher rates when the interest rate in the economy is high in order to compensate for the weakened threat of repossession when the interest rate in the economy is low. Also, unless the borrower's balance is sufficiently close to the preferential debt limit, the borrower's enjoys an increase of the negative amortization limit when the market rate switches to low and vice versa.

Figure 3 presents the variable interest rate charged on the balance of the optimal option ARM above the preferential debt limit in the parametrized environment of Section 4.3.

5.2 Alternative Implementations

In this section, we discuss alternative implementations of the optimal allocation. Option ARM, which is essentially a single revolving line of credit, is one but not the only way to implement the optimal allocation. Its main advantage is that evolution of the balance on the loan is determined entirely by the borrower's payments and the interest rates charged on the balance. We show that the optimal allocation can be also implemented using interest only mortgages with home equity line of credit (HELOC), a contract that emerges from the option ARM by explicitly separating the interest payments on the mortgage debt from the embedded credit line.

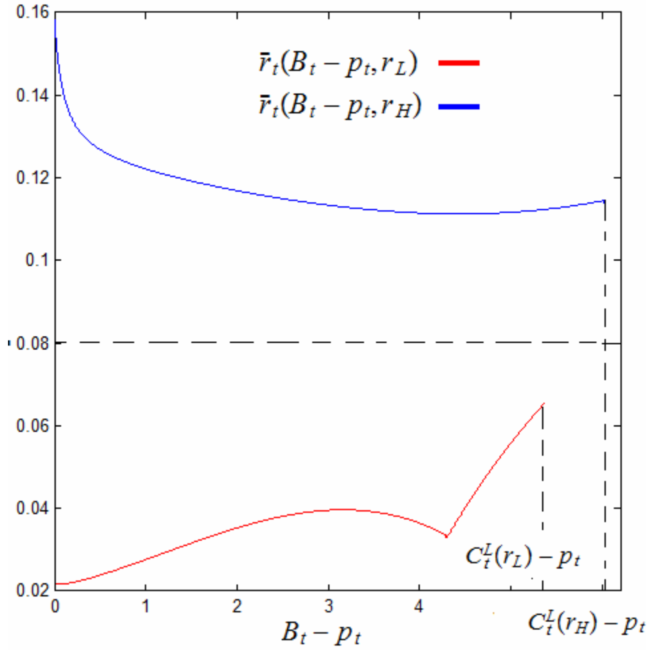
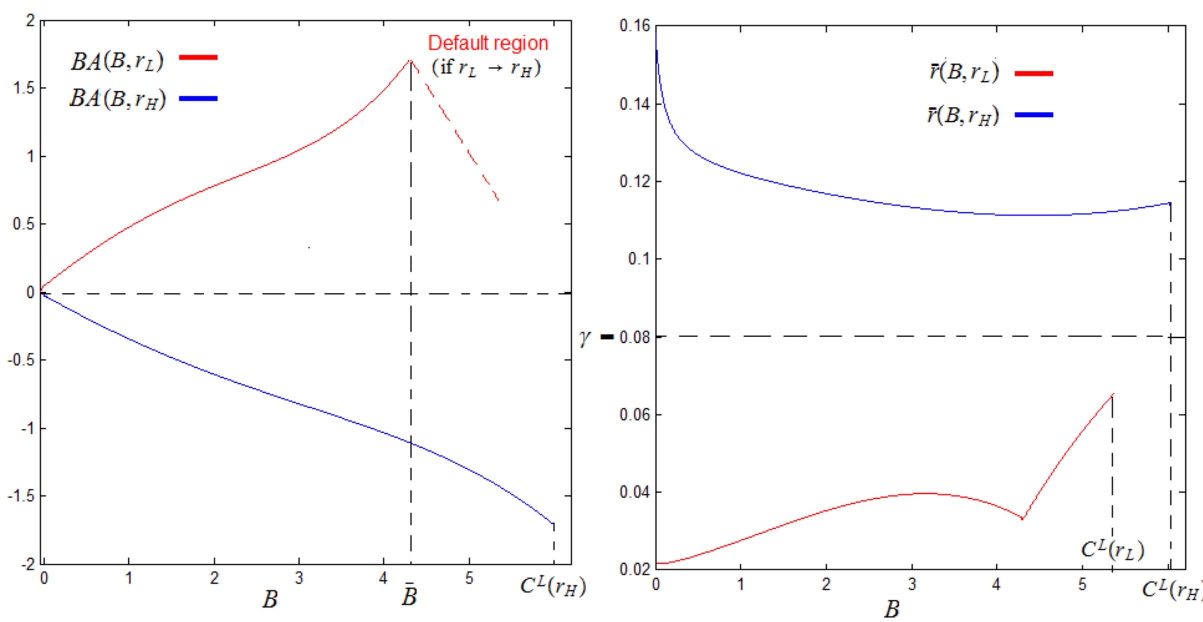


Figure 3: Variable interest rate charged on the optimal option ARM's balance above the preferential debt limit.

Figure 4: Optimal balance adjustment and the variable interest rate on the HELOC debt.



An interest only mortgage with HELOC is a combination of two forms of debt - an interest only mortgage and a second "piggyback"[17] mortgage that closes simultaneously with the first. Recently, there has been a noticeable increase in the use of "piggyback" mortgages, and many lenders structure a second "piggyback" loan as a home equity line of credit. These lines are revolving lines of credit like credit cards, yet they are secured by the borrower's home collateral. Homeowners who pay off the line of credit can continue to draw upon it and use the funds for other purposes.

5.2.1 Interest Only Mortgage with HELOC and Two Way Balance Adjustment

Unlike the option ARM, this implementation does not make use of the preferential debt but instead allows for adjustments of the HELOC balance when the interest rate in the economy changes. A definition below formally describes an interest only mortgage with HELOC and two way balance adjustment.

Definition 8 *An interest only mortgage with HELOC and two way balance adjustment consists of:*

- *interest only mortgage with a required coupon (interest payment)* x_t,
- *HELOC with interest rate* $\bar{r}_t$ *charged on HELOC balance* B_t *and credit limit* C_t^L,
- *Adjustment* BA_t *of HELOC balance, applicable whenever the interest rate in the economy changes,*
- *Default happens when either the coupon is not paid or the HELOC balance exceeds the credit limit, in which case the lender repossesses the home.*

The following proposition shows that the optimal allocation can be implemented using an interest only mortgage with HELOC and two way balance adjustment.

Proposition 6 *There exists an optimal interest only mortgage with HELOC and two way balance adjustment that has the following properties:*

$$\bar{r}_t(B_t, r_t) = \gamma + \delta(r_t)\frac{\left[\psi(a^1(r_t) - B_t, r_t) - \psi(a^1(r_t), r_t)\right]}{B_t}, \tag{22}$$

$$C_t^L(r_t) = a^1(r_t) - A, \tag{23}$$

$$x_t(r_t) = \theta + \mu - \gamma a^1(r_t) + \delta(r_t)\psi(a^1(r_t), r_t), \tag{24}$$

$$BA(B_t, r_t) = -\psi(a^1(r_t) - B_t, r_t) + (a^1(r_t^c) - a^1(r_t)). \tag{25}$$

Under the terms of this mortgage, it is optimal for the borrower to use all available cash flows to pay down the HELOC balance, and consumes all excess cash flows once the HELOC balance becomes zero. The borrower's

[17] Named as such in the housing finance industry because a second mortgage is "piggybacked" onto the original mortgage loan.

expected payoff, a_t, is determined by the HELOC balance as follows:

$$a_t = A + \left[C_t^L(r_t) - B_t\right] = a^1(r_t) - B_t. \tag{26}$$

Proof In the Appendix. ■

Similarly to loan balance in the option ARM implementation, the HELOC balance plays the role of a state variable that summarizes all the relevant information about the past and tracks the borrower's continuation utility according to equation (26). The interest rate along with the required mortgage coupon payment, balance adjustment, and the credit line limit, determine the dynamics of the HELOC balance and default time. Unlike the option ARM, the interest only mortgage with HELOC and two way balance adjustment does not have preferential debt feature. Instead, it has balance adjustment. Similarly to the adjustments in size of preferential debt, balance adjustments, which happen when the interest rate in the economy changes, help to implement the effects that the interest rate in the economy has on the borrower's continuation utility under the optimal contract.

Proposition 6, together with Lemma 3, implies

Corollary 3 *The optimal interest only mortgage with HELOC and two way balance adjustment has the following properties:*

i) Let $\bar{B} = a^1(r_L) - \bar{a}$ where $\bar{a}$ is defined in Lemma 3. Then, whenever $B_t \in [\bar{B}, C_t^L(r_L))$, an instantaneous change of the interest rate in the economy from r_L to r_H triggers default on the mortgage;

ii) $BA(B, r_t) = 0$ for $B = 0$. Suppose further that the optimal function ψ satisfies the properties of Condition 1. Then,

- *$BA(B, r_L)$ is positive and strictly increasing in B for $B \in (0, \bar{B}]$,*
- *$BA(B, r_H)$ is negative and strictly decreasing in B for $B \in (0, C_t^L(r_H)]$,*
- *$\bar{r}_t(B', r_L) < \gamma < \bar{r}_t(B'', r_H)$, for any $B' \in [0, C_t^L(r_L)]$, $B'' \in [0, C_t^L(r_H)]$.*

As the above corollary shows, under the optimal interest only mortgage with HELOC and two way balance adjustment, whenever the HELOC balance is close to the credit limit, an increase in the interest rate in the economy would result in default on the mortgage. The variable interest rate on the HELOC balance positively correlates with the interest rate in the economy. Figure 4 presents the optimal balance adjustment and the variable interest rate charged on the HELOC balance in the parametrized environment of Section 4.3.

Although mortgages with HELOC and two way balance adjustment are interesting from the theoretical point of view, we do not yet observe anything like that in practice. While we actually observe reductions of mortgage debt balance in the form of "cramdown" provisions, the unusual feature of these mortgages is the automatic increase in debt balance in response to a market interest rate increase. Below we discuss an implementation using the interest only mortgage with HELOC with a preferential rate and one way balance adjustment that addresses this issue.

5.2.2 Interest Only Mortgage with HELOC with Preferential Rate and One Way Balance Adjustment

In this section, we consider a combination of an interest only mortgage with HELOC, where a part of the HELOC balance is subject to a preferential interest rate. Only reductions of the HELOC balance are allowed. The definition below provides a formal description of this class of mortgage contracts.

Definition 9 *An interest only mortgage with HELOC with preferential rate and one way balance adjustment consists of:*

- *Interest only mortgage with a required coupon (interest payment) x_t,*
- *HELOC with interest rate $\bar{r}_t^p$ charged on the portion of the HELOC balance B_t below p_t and interest rate $\bar{r}_t$ charged on the portion of the balance above p_t, and credit limit C_t^L.*
- *Reductions BA_t^- in the HELOC balance, applicable whenever the interest rate in the economy decreases.*
- *Default happens when either the coupon is not paid or the HELOC balance exceeds the credit limit, in which case the lender repossesses the home.*

The proposition below shows that the optimal allocation can be implemented using an interest only mortgage with HELOC with preferential rate and one way balance adjustment.

Proposition 7 *There exists an optimal interest only mortgage with HELOC with preferential rate and one way balance adjustment that has the following features:*

$$\bar{r}_t^p = 0, \tag{27}$$

$$\bar{r}_t(B_t - p_t, r_t) = \gamma + \delta(r_t)\frac{\left[\psi(a^1(r_t) - (B_t - p_t), r_t) - \psi(a^1(r_t), r_t)\right]}{B_t - p_t}, \tag{28}$$

$$dp_t = \begin{cases} \left[\psi(a^1(r_L) - (B_t - p_t), r_L) - \left(a^1(r_H) - a^1(r_L)\right)\right] I_{(r_{t\_}=r_L)} \ if \ B_t \geq p_t \\ 0, \ if \ B_t < p_t \end{cases}, \tag{29}$$

$$BA^-(B_t - p_t) = -\psi(a^1(r_H) - (B_t - p_t), r_H) + \left[a^1(r_L) - a^1(r_H)\right], \tag{30}$$

$$x_t(r_t) = \theta + \mu - \gamma a^1(r_t) + \delta(r_t)\psi(a^1(r_t), r_t), \tag{31}$$

$$C_t^L(p_t, r_t) = p_t + a^1(r_t) - A. \tag{32}$$

Under the terms of this mortgage, it is optimal for the borrower to use all available cash flows to pay down HELOC balance B_t when $B_t > p_t$, and consumes all excess cash flows once the balance drops to p_t. For *balance $B_t \geq p_t$, the borrower's continuation utility a_t is equal to:*

$$a_t = A + \left[C_t^L(p_t, r_t) - B_t\right] = a^1(r_t) - (B_t - p_t). \tag{33}$$

If the amount of debt subject to the preferential rate falls to zero, the mortgage is reset to a more complicated contract that implements the continuation of the optimal allocation.

Proof In the Appendix. ■

The above implementation combines such features as preferential debt, which is present in option ARM, and balance adjustment, which is present in an interest only mortgage with HELOC with preferential rate and two way balance adjustment. Both these features are used to implement adjustments in the borrower's continuation utility due to changes of the market interest rate. Unlike the implementation with the interest only mortgage and HELOC with two way balance adjustment, this implementation avoids increasing the borrower's balance when the interest rate in the economy changes from low to high by decreasing instead the amount of the HELOC balance subject to the preferential rate.

Proposition 7, together with Lemma 3, imply

Corollary 4 *The optimal interest only mortgage with HELOC with preferential rate and one way balance adjustment has the following properties:*

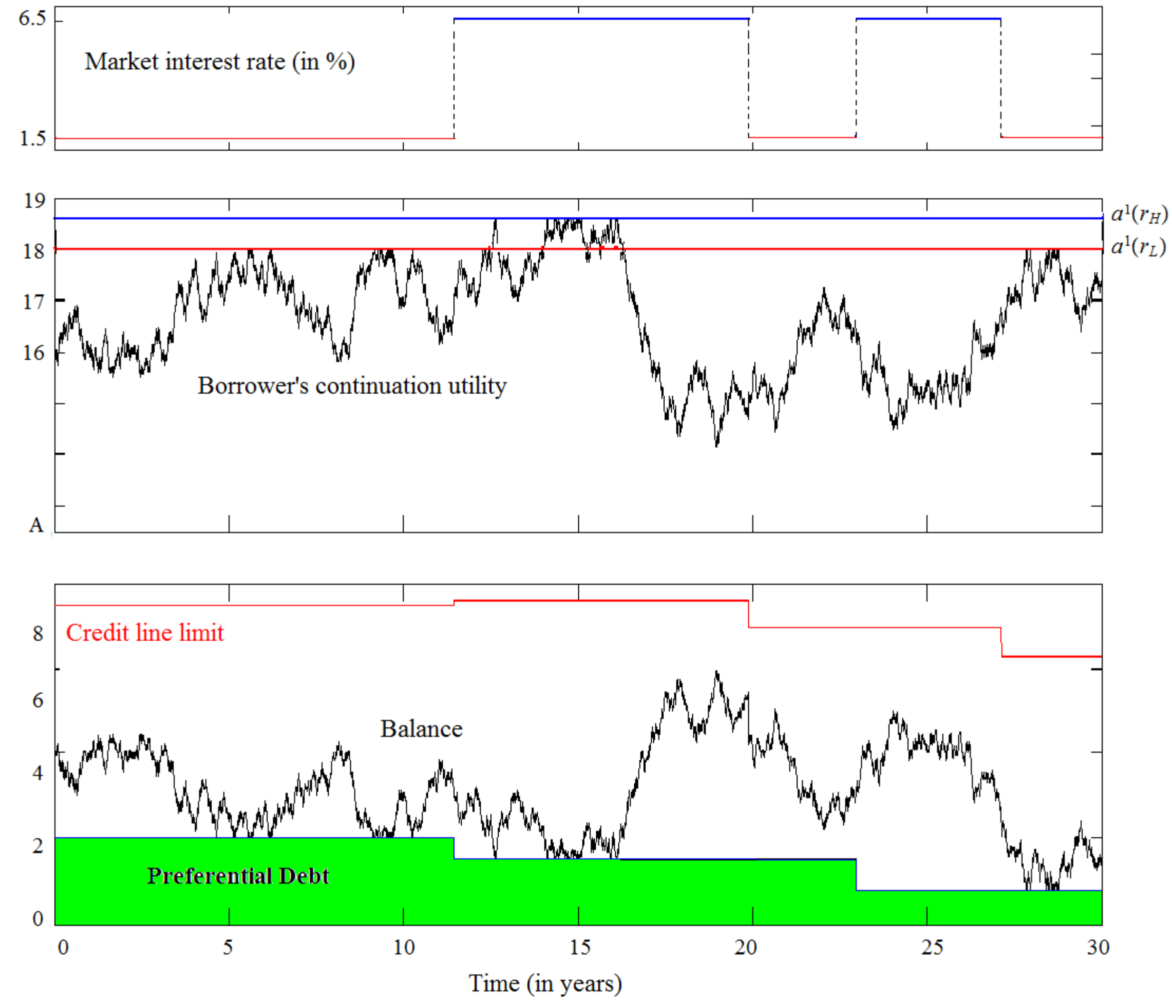
i) Let $\bar{B}_t = p_t + a^1(r_L) - \bar{a}$ where $\bar{a}$ is defined as in Lemma 3. Then, whenever $B_t \in [\bar{B}_t, C_t^L(p_t, r_L))$, an instantaneous increase in the interest rate in the economy triggers default on the mortgage;

ii) $BA^-(B_t - p_t) = 0$ for $B_t = p_t$. Suppose further that the optimal function ψ satisfies the properties of Condition 1. Then,

- *$BA^-(B_t - p_t)$ is negative and strictly decreasing in $(B_t - p_t)$ for $B_t \in (p_t, C_t^L(r_H)]$,*
- *$dp_t < 0$ for any $B_t > p_t$, whenever the interest rate in the economy increases,*
- *$\bar{r}_t(B' - p_t, r_L) < \gamma < \bar{r}_t(B'' - p_t, r_H)$, for any $B' \in [p_t, C_t^L(r_L)]$, $B'' \in [p_t, C_t^L(r_H)]$.*

As the above corollary shows, whenever the HELOC balance is close to the credit limit, an increase in the interest rate would cause the liquidation of the mortgage. If the optimal adjustment function ψ satisfies the properties of Condition 1, the variable interest rate on the HELOC balance (28) positively correlates with the interest rate in the economy. A decrease in the interest rate in the economy causes a reduction of the borrower's HELOC balance. This reduction can be interpreted as offering the borrower an automatic "cramdown" provision, whenever the interest rate in the economy goes down. An increase in the interest rate in the economy causes a drop in the amount of debt subject to the preferential interest rate. Consequently, under this contract, it is optimal to reduce the preferential treatment of the HELOC debt over time. We

Figure 5: A simulated path of the optimal interest only mortgage with HELOC.



note that a declining preferential treatment of debt over time is a typical feature of many mortgage contracts currently offered in the housing finance market.

The top part of Figure 5 presents a simulated path of the market interest rate, the middle one presents a simulated path of the borrower's continuation utility under the optimal contract, and the bottom one presents the behavior of credit line, the preferential debt range, and the HELOC balance implied by the optimal mortgage contract of Proposition 7, where the parameters of the model are set as in Section 4.3. Figure 6 presents the optimal negative balance adjustment and the variable interest rate on the HELOC debt in this parametrized example.

6 Approximate Implementations

In this section we consider simpler mortgage contracts that implement the optimal allocation approximately. The idea of approximate implementation is based on the observation that the results of Propositions 5 - 7 concerning the implementation of the optimal allocation do not rely on any particular properties of functions ψ and a^1. In other words, if we replace the optimal function ψ from Proposition 3 by any function $\hat{\psi} : [A, \infty) \times \{r_L, r_H\} \to R$, such that, $\hat{\psi}(a, r) + a \geq A$ for any $a \geq A$, and replace the reflection barriers $a^1(r)$ by any finite $\hat{a}^1(r) \geq A$, then the resulting suboptimal contract will remain incentive compatible.

In what follows, we will focus on the following approximation of the optimal functions ψ and a^1.

Definition 10 *The approximately optimal function $\hat{\psi}$ and $\hat{a}^1$ satisfy*

- $\hat{a}^1 \equiv \hat{a}^1(r_L) = \hat{a}^1(r_H) = \inf\{a \geq A : \ \psi(a, r_L) = \psi(a, r_H) = 0\}$,
- $\hat{\psi}(a, r_L) = \begin{cases} A - a & \text{for } a \in [A, \bar{a}] \\ -\left(\frac{\bar{a}-A}{\hat{a}^1-\bar{a}}\right)(\hat{a}^1 - a) & \text{for } a \in [\bar{a}, \hat{a}^1] \end{cases}$,
- $\hat{\psi}(a, r_H) = \left(\frac{\bar{a}-A}{\hat{a}^1-A}\right)(\hat{a}^1 - a) \quad \text{for } a \in [A, \hat{a}^1]$,

  where ψ and a^1 are the functions from the optimal contract of Proposition 3.

Figure 7 presents the approximately optimal functions $\hat{\psi}$ and $\hat{a}^1$, together with their optimal counterparts in the parametrized environment of Section 4.3. We note that function $\hat{\psi}$ satisfies Condition 1. We also note that $\hat{\psi}$ approximates ψ reasonably well except for the region with high continuation utility, which is much less important in terms of contract optimality than the region near the bankruptcy threshold A. As we will show next, this approximation leads to simpler mortgage contracts that do not sacrifice much in terms of their payoff efficiency.

6.1 Approximately Optimal Option ARM with Preferential Rate

In Section 5.1, we demonstrated that the optimal allocation can be implemented using the optimal option ARM. Replacing optimal function ψ with approximately optimal function $\hat{\psi}$ defined in Section 6 gives us an approximately optimal option ARM with the following parameters

Figure 6: The optimal negative balance adjustment and the variable interest rate on the HELOC debt.

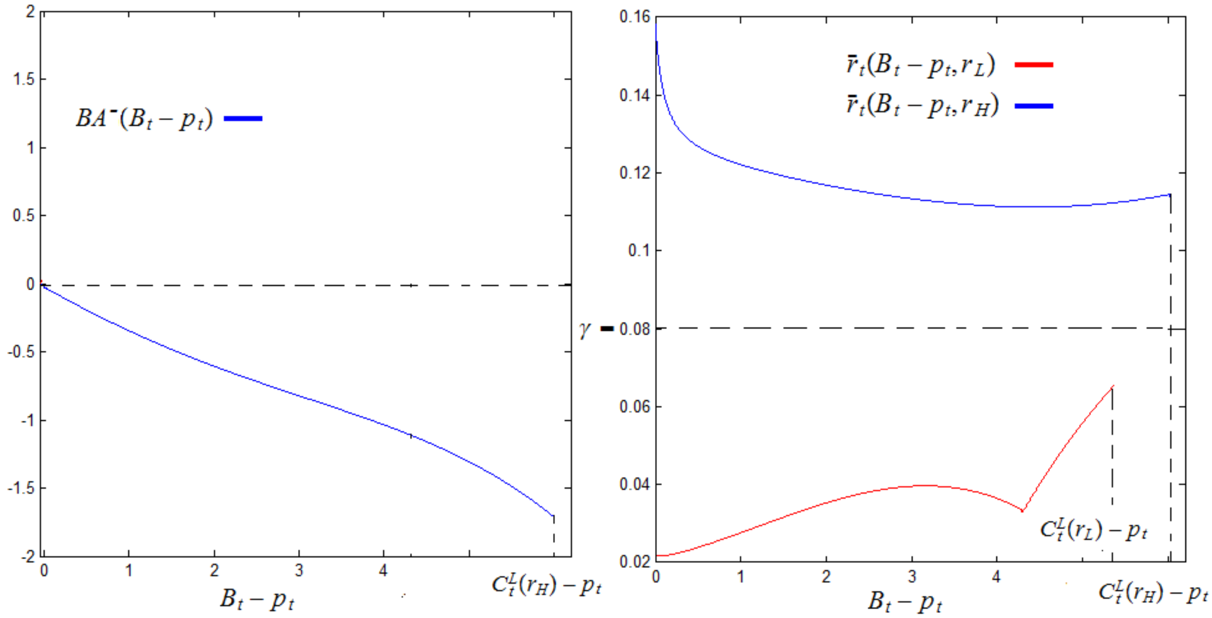
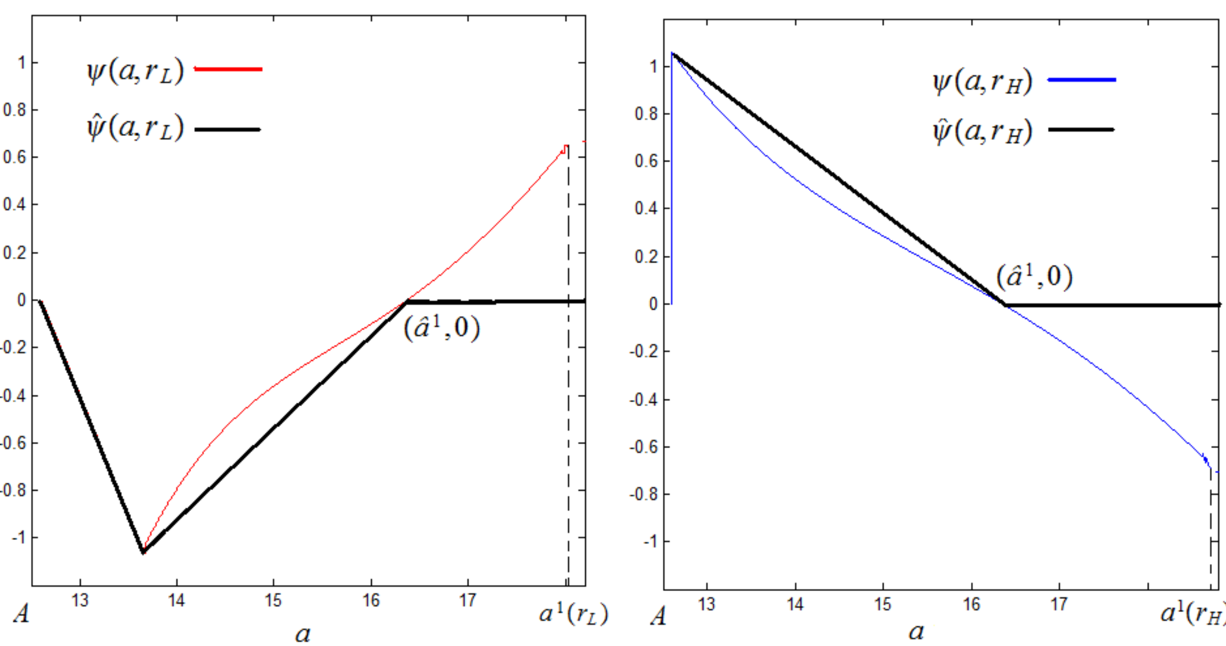


Figure 7: The approximately optimal function $\hat{\psi}$ and $\hat{a}^1$.



$$C_t^L(p_t) = p_t + \hat{a}^1 - A \tag{34}$$

$$\bar{r}_t^p(p_t) = \frac{\theta + \mu - \gamma\hat{a}^1}{p_t} \tag{35}$$

$$\bar{r}_t(B_t - p_t, r_t) = \begin{cases} \gamma - \delta(r_L)\left(\frac{\bar{a}-A}{\hat{a}^1-\bar{a}}\right) & \text{for } B_t \in [p_t, \hat{B}_t] \quad \text{and } r_t = r_L \\ \gamma - \delta(r_L)\left(\frac{\hat{a}^1-A-B_t+p_t}{B_t-p_t}\right) & \text{for } B_t \in [\hat{B}_t, C_t^L] \quad \text{and } r_t = r_L \\ \gamma + \delta(r_H)\left(\frac{\bar{a}-A}{\hat{a}^1-A}\right) & \text{for } B_t \in [p_t, C_t^L] \quad \text{and } r_t = r_H \end{cases}, \tag{36}$$

$$dp_t = \begin{cases} -\left(\frac{\bar{a}-A}{\hat{a}^1-\bar{a}}\right)(B_t - p_t) & \text{for } B_t \in [p_t, \hat{B}] \quad \text{and } r_t = r_L \\ \left(\frac{\bar{a}-A}{\hat{a}^1-A}\right)(B_t - p_t) & \text{for } B_t \in [p_t, C_t^L] \quad \text{and } r_t = r_H \\ 0 & \text{for } B_t < p_t \end{cases}, \tag{37}$$

where $\hat{B}_t = p_t + \hat{a}^1 - \bar{a}$.

Proposition 8 *Under the terms of option ARM given by (34-37), it is optimal for the borrower to use all available cash flows to pay down balance B_t when $B_t > p_t$, and consumes all excess cash flows once the balance drops to p_t. The borrower's continuation utility $\hat{a}_t$ is determined by the balance above the preferential debt limit as follows:*

$$\hat{a}_t = A + C_t^L - B_t = \hat{a}^1 - (B_t - p_t) \tag{38}$$

Proof In the Appendix. ■

The intuition behind incentive compatibility of the postulated strategy of the borrower under the above mortgage contract is the same as in the case of the optimal mortgage contract of Proposition 5. The negative amortization limit (34), the interest rates (35) and (36), and the preferential debt limit (37) play the same role in the approximate implementation as their counterparts from Proposition 5 in the exact implementation of the optimal allocation.

As the following corollary shows, a decrease in the interest rate in the economy results in smaller mortgage interest rates, higher amount of debt subject to the preferential rate, and higher negative amortization limit.

Corollary 5 *The approximately optimal option ARM has the following properties:*

(i) $\bar{r}_t(B' - p_t, r_L) < \gamma < \bar{r}_t(B'' - p_t, r_H)$, *for any* $B' \in [p_t, C_t^L(r_L)]$, $B'' \in [p_t, C_t^L(r_H)]$.

(ii) $dp_t < 0$, $dC_t^L < 0$, $d\bar{r}_t^p > 0$ *whenever* r_t *increases and* $B_t > p_t$,

$dp_t > 0$, $dC_t^L > 0$, $d\bar{r}_t^p < 0$ *whenever* r_t *decreases and* $B_t > p_t$,

Proof follows directly from (34) - (37). ■

Figure 8 shows the interest rate charged on the balance of the approximate option ARM above the preferential debt limit.

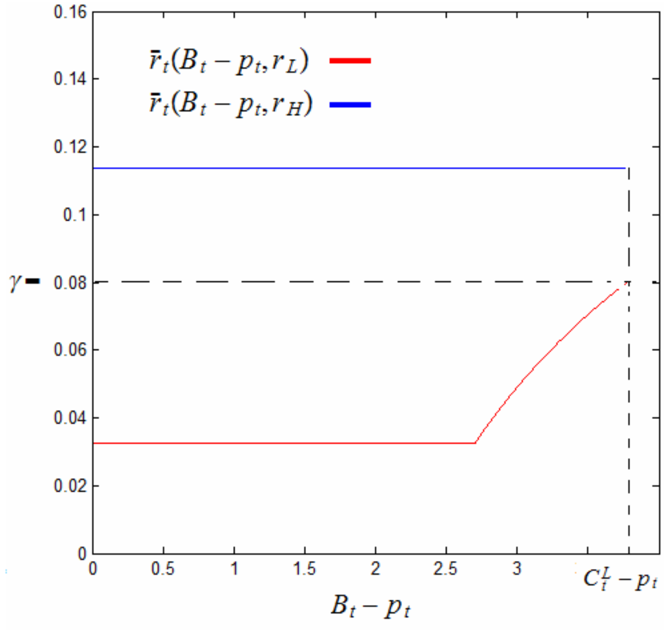


Figure 8: Variable interest rate charged on the approximately optimal option ARM's balance above the preferential debt limit.

6.2 Approximately Optimal Interest Only FRM with HELOC and Two Way Balance Adjustment

In Section 5.2.1, we demonstrated that the optimal allocation can be implemented using the optimal interest only mortgage with HELOC and two way balance adjustment. Replacing optimal function ψ with approximately optimal function $\hat{\psi}$ defined in Section 6 gives us an approximately optimal interest only FRM with HELOC and two way balance adjustment with the following parameters:

$$x_t = x = \theta + \mu - \gamma \hat{a}^1, \tag{39}$$

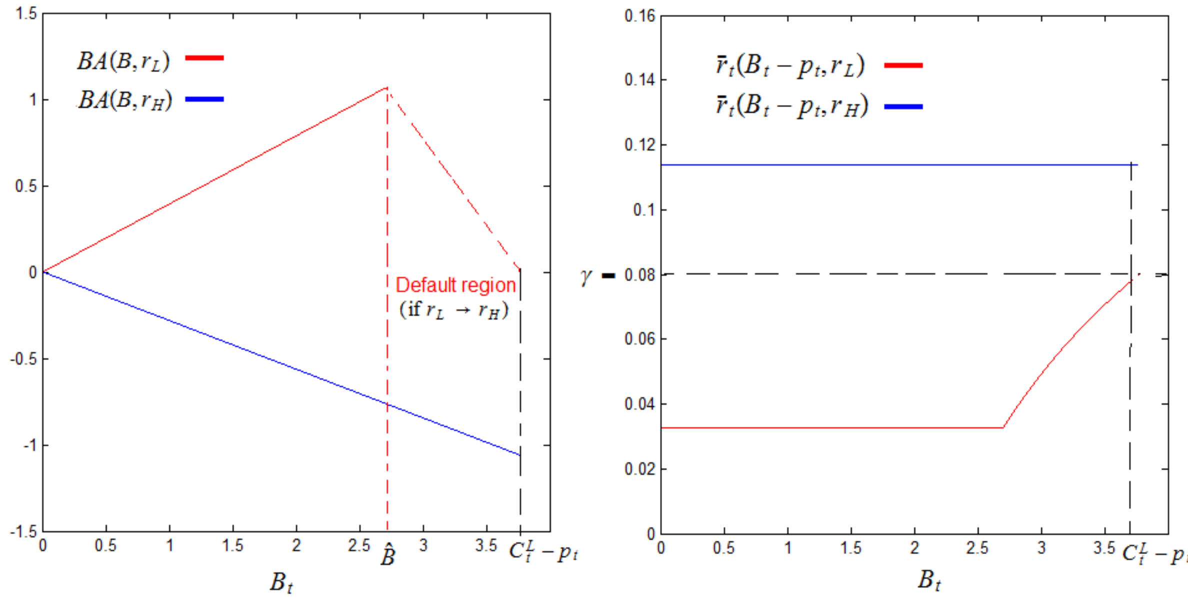
$$C_t^L = C^L = \hat{a}^1 - A, \tag{40}$$

$$\bar{r}_t(B_t, r_t) = \begin{cases} \gamma - \delta(r_L)\left(\frac{\bar{a}-A}{\hat{a}^1-\bar{a}}\right) & \text{for } B_t \in [0, \hat{B}] & \text{and } r_t = r_L \\ \gamma - \delta(r_L)\left(\frac{\hat{a}^1-A-B_t}{B_t}\right) & \text{for } B_t \in [\hat{B}, C^L] & \text{and } r_t = r_L \\ \gamma + \delta(r_H)\left(\frac{\bar{a}-A}{\hat{a}^1-A}\right) & \text{for } B_t \in [0, C^L] & \text{and } r_t = r_H \end{cases}, \tag{41}$$

$$BA(B_t, r_t) = \begin{cases} \left(\frac{\bar{a}-A}{\hat{a}^1-\bar{a}}\right) B_t & \text{for } B_t \in [0, \hat{B}] & \text{and } r_t = r_L \\ -\left(\frac{\bar{a}-A}{\hat{a}^1-A}\right) B_t & \text{for } B_t \in [0, C^L] & \text{and } r_t = r_H \end{cases}, \tag{42}$$

where $\hat{B} = \hat{a}^1 - \bar{a}$.

Figure 9: Approximately optimal balance adjustment and the variable interest rate on the HELOC debt.



Proposition 9 *Under the interest only FRM with HELOC and two way balance adjustment with the parameters given by (39) - (42), it is optimal for the borrower to use all available cash flows to pay down the HELOC balance, and consume all excess cash flows once the HELOC balance becomes zero. The borrower's continuation utility a_t is determined by the HELOC balance as follows:*

$$\hat{a}_t = A + \left[C_t^L(r_t) - B_t\right] = \hat{a}^1 - B_t. \tag{43}$$

Proof follows from the proof of Proposition 6 by replacing function ψ with $\hat{\psi}$ and the reflection barriers $a^1(r)$ with $\hat{a}^1$. ■

The approximately optimal interest only mortgage with HELOC takes a simple form of the interest only FRM with the constant interest coupon payment of (39). The HELOC has a constant credit limit given by (40), and a simple variable rate given by (41). It follows from (39) - (42) that this mortgage has the following properties.

Corollary 6 *The approximately optimal interest only FRM with HELOC and two way balance adjustment has the following properties:*

i) $BA(B, r_t) = 0$ *for* $B = 0$, *and*

- $BA(B, r_L)$ *is positive and strictly increasing in* B *for* $B \in (0, \hat{B}]$,

– $BA(B, r_H)$ *is negative and strictly decreasing in* B *for* $B \in (0, C^L]$,

(ii) $\bar{r}_t(B', r_L) < \gamma < \bar{r}_t(B'', r_H)$, *for any* $B' \in [0, C^L]$, $B'' \in [0, C^L]$.

As the above corollary shows, under the approximately optimal interest only FRM with HELOC and two way balance adjustment, a decrease in the interest rate in the economy causes a decrease in the borrower's HELOC balance and vice versa. The magnitude of these adjustments is linearly proportional to the HELOC balance. The variable interest rate on the HELOC balance positively correlates with the lender's interest rate, and does not depend on the borrower's HELOC balance, except the debt region $[\hat{B}, C^L]$, where the HELOC interest rate increases with the balance when the interest rate in the economy is low ($r_t = r_L$). Figure 9 presents the approximately optimal balance adjustments and the variable interest rate on the HELOC debt in the parametrized environment of Section 4.3.

6.3 Approximately Optimal Interest Only FRM with HELOC with Preferential Interest Rate and One Way Balance Adjustment

In Section 5.2.2, we showed that the optimal allocation can be implemented using the optimal interest only mortgage with HELOC and one way balance adjustment. Replacing optimal function ψ with approximately optimal function $\hat{\psi}$ defined in Section 6 gives us an approximately optimal interest only FRM with HELOC and one way balance adjustment with the following parameters:

$$x_t = x = \theta + \mu - \gamma \hat{a}^1, \tag{44}$$

$$C_t^L(p_t) = p_t + \hat{a}^1 - A, \tag{45}$$

$$\bar{r}_t^p = 0, \tag{46}$$

$$\bar{r}_t(B_t - p_t, r_t) = \begin{cases} \gamma - \delta(r_L)\left(\frac{\bar{a}-A}{\hat{a}^1-\bar{a}}\right) & \text{for } B_t \in [p_t, \hat{B}_t] \quad \text{and } r_t = r_L \\ \gamma - \delta(r_L)\left(\frac{\hat{a}^1-A-B_t+p_t}{B_t-p_t}\right) & \text{for } B_t \in [\hat{B}_t, C_t^L] \quad \text{and } r_t = r_L \\ \gamma + \delta(r_H)\left(\frac{\bar{a}-A}{\hat{a}^1-A}\right) & \text{for } B_t \in [p_t, C_t^L] \quad \text{and } r_t = r_H \end{cases}, \tag{47}$$

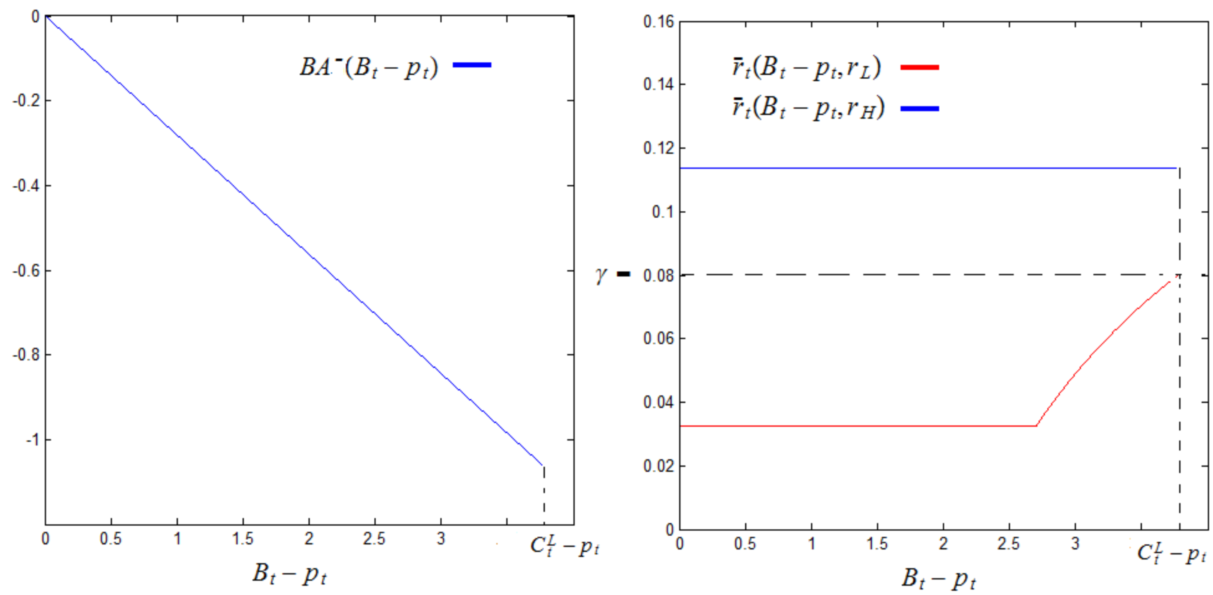
$$dp_t = \begin{cases} -\left(\frac{\bar{a}-A}{\hat{a}^1-\bar{a}}\right)(B_t - p_t) & \text{for } B_t \in [p_t, \hat{B}_t] \quad \text{and } r_t = r_L \\ 0 & \text{for } B_t < p_t \end{cases}, \tag{48}$$

$$BA^-(B_t - p_t) = -\left(\frac{\bar{a} - A}{\hat{a}^1 - A}\right)(B_t - p_t), \tag{49}$$

where $\hat{B}_t = p_t + \hat{a}^1 - \bar{a}$.

Proposition 10 *Under the interest only FRM with HELOC and one way balance adjustment with the parameters given by (44) - (49), it is optimal for the borrower to pay down HELOC balance* B_t *when* $B_t > p_t$, *and*

Figure 10: The approximately optimal negative balance adjustment and the interest rate on the HELOC debt.



consumes all excess cash flows once the balance drops to p_t. For balance $B_t \geq p_t$, the borrower's continuation utility a_t is equal to

$$a_t = A + \left[C_t^L(p_t, r_t) - B_t\right] = \hat{a}^1 - (B_t - p_t). \tag{50}$$

Proof follows from the proof of Proposition 7 by replacing function ψ with $\hat{\psi}$ and the reflection barriers $a^1(r)$ with $\hat{a}^1$. ■

The approximately optimal interest only mortgage with HELOC with preferential interest rate and one way balance adjustment takes the simple form of the interest only FRM, with the constant interest coupon payment of (44), combined with the HELOC with the credit limit of (45) and the simple variable rate given by (46).

Corollary 7 *The approximately optimal interest only FRM with HELOC with preferential rate and one way balance adjustment has the following properties:*

i) $BA^{-}(B_t - p_t) = 0$ *for* $B_t = p_t$*, and*

- $BA^{-}(B_t - p_t)$ *is negative and strictly decreasing in* $(B_t - p_t)$ *for* $B_t \in (p_t, C_t^L(r_H)]$,
- $dp_t \leq 0$, $dC_t^L \leq 0$ *for any* $B_t \geq p_t$*, with strict inequality whenever the interest rate,* r_t*, increases and* $B_t > p_t$,

(ii) $\bar{r}_t(B' - p_t, r_L) < \gamma < \bar{r}_t(B'' - p_t, r_H)$*, for any* $B' \in [p_t, C_t^L(r_L)]$*,* $B'' \in [p_t, C_t^L(r_H)]$.

As the above corollary shows, a decrease in the interest rate in the economy causes a decrease in the borrower's HELOC balance. The magnitude of this adjustment is linearly proportional to the HELOC balance, and, as before, can be interpreted as offering the borrower an automatic "cramdown" provision. An increase in the interest rate in the economy causes a drop in the amount of debt subject to the preferential interest rate and a decrease in the credit line limit. Consequently, under this contract, the preferential HELOC debt treatment is reduced over time. The variable interest rate charged on the HELOC balance positively correlates with the interest rate in the economy, and does not depend on the borrower's debt balance, except the debt region $[\hat{B}, C^L]$, where it increases with the balance when that the interest rate in the economy is low.

Figure 10 presents the negative balance adjustment and the variable interest rate on the HELOC debt as a function of the borrower's mortgage debt above the preferential range in the parametrized environment of Section 4.3.

7 Efficiency Gains Due to Optimal and Approximately Optimal Contracts

What are the efficiency gains from using mortgages that allow for the adjustable rate and the preferential debt treatment? How close, in terms of efficiency, is the approximately optimal mortgage contract to the optimal one? To study these questions, we compare the lender's value under, respectively, the approximately optimal contract and the optimal contract, with a best value achievable for the lender, for a given borrower's continuation utility, under a simpler mortgage[18] with fixed rate and no preferential debt.

Figure 11 presents the percentage improvement (in basis points) in the lender's value[19] across the initial continuation utility of the borrower under, respectively, the approximately optimal contract and the optimal contract, relative to the highest value of the lender achievable under a simpler mortgage with fixed rate and no preferential debt. The computations are performed in the parametrized environment of Section 4.3.

As we observe from Figure 11, the value of the lender under the approximately optimal contract is close to that under the optimal contract with loss ranging from zero to just above 10 basis points of the value. Both contracts yield much better performance compared to a simpler mortgage with fixed rate and no preferential debt. The gain can be as high as 70 basis points of the lender's value and, in the renegotiation proof region, the gain can be close to above 50 basis points for the optimal contract and close to 40 basis points for the approximately optimal contract. Given that the origination fee charges in mortgages are usually in the order

[18] This mortage implements an allocation found by solving, for a given continuation utility of the borrower, the problem of maximizing the lender's expected profit subject to incentive compatibility and promise keeping constraints, and subject to an additional constraint that forbids any adjustments in the borrower's continuation value due to changes in the lender's interest rate.

[19] We remember that the lender's value represent his expected profit plus the loan amount to the borrower $[P - Y_0]$. Therefore, with no downpayment ($Y_0 = 0$) the lender breaks even if his value is at least as large as the price of home.

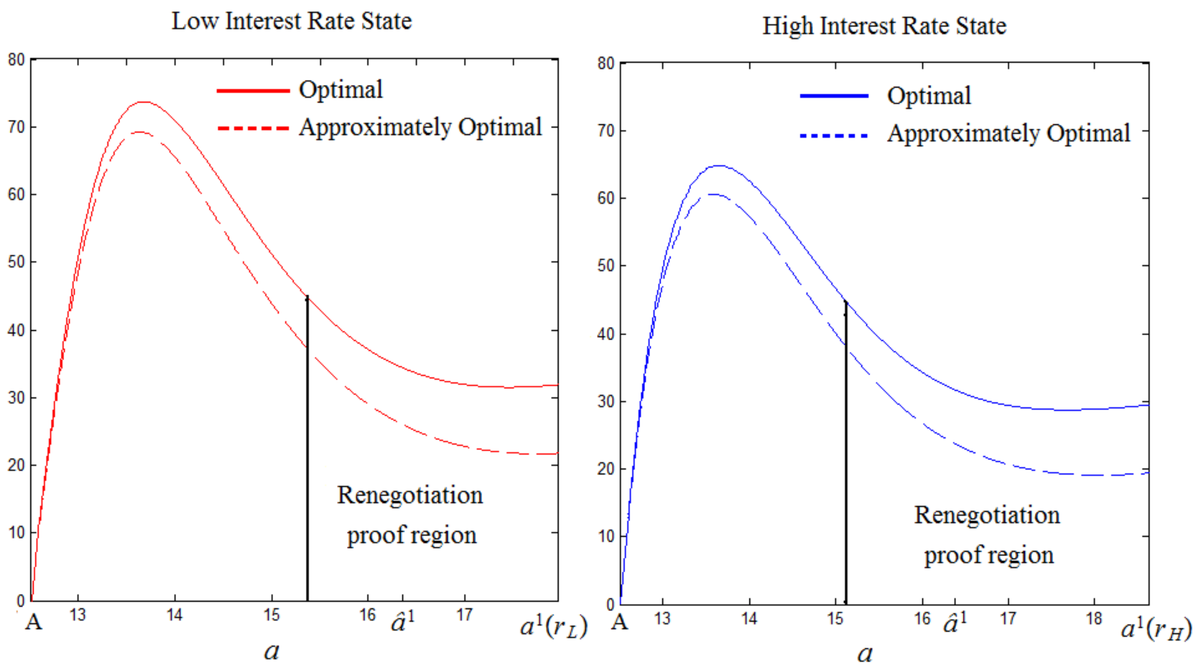


Figure 11: Gains (in basis points) of the lender's value under the optimal and the approximately optimal contract relative to a mortgage with fixed rate and no preferential debt.

of 0.5% to 1%[20] of the loan amount these gains are very substantial.

Many reasonable models of determination of initial starting point in terms of the borrower's continuation utility will have a property that the borrower's continuation utility increases with the amount of downpayment and the borrower's future expected income. Figure 11 indicates that, if this is the case, the largest efficiency gains in the renegotiation proof region are to be realized on the optimal mortgages given to those households who buy pricey homes given their income level or make little or no downpayment. Thus, our analysis provides theoretical evidence that high concentration of the alternative mortgages in the subprime market can be economically efficient. Our comparison also suggests that the mortgage terms can be considerably simplified, with little loss of efficiency, by implementing the approximately optimal allocation instead of the optimal one.[21]

8 Concluding Remarks

Recent years have seen a rapid growth in originations of more sophisticated alternative mortgage products (AMPs), such as option adjustable rate mortgages (option ARMs) and interest only mortgages. Critics

[20] See for example Freddie Mac (2007).
[21] These results hold across all parameterizations we considered.

of AMPs point out that they seem to be more profitable for lenders than traditional mortgages. They conclude that AMPs allow lenders to profiteer at the expense of homeowners. However, this paper shows that the properties of AMPs are consistent with the properties of the optimal allocation governing the relationship between the borrower and the lender, which represents a Pareto improvement over traditional mortgages. As a consequence, it is possible that both lenders and borrowers can benefit from AMPs. Critics of AMPs have raised the concern that teaser rates and low minimum payments can result in substantially higher mortgage payments and, as a consequence, higher default rates when the market interest rate increases. Nevertheless, this paper demonstrates that this does not necessarily contradict the optimality of AMPs. Under the optimal mortgage contract, mortgage payments and default rates are indeed higher when the market interest rate is high. However, borrowers benefit from low mortgage payments and low default rates when the market interest rate is low.

The optimal contracts do not allow borrowers to refinance their mortgages with another lender. Offering this option would increase the borrower's reservation value, which would limit the ability to provide him incentives to repay his debt, resulting in a decrease of efficiency of the contract. Therefore, our results lend support to prepayment penalties on refinancing. Introduction of borrowers' mobility would result in a "soft" prepayment penalty, borrowers could sell their home at anytime without penalty, but if they want to refinance the mortgage, they would pay the prepayment penalty sufficiently large to discourage it.

In this paper, we ignored inflation, which is an important consideration for home buyers choosing between ARMs and FRMs.[22] However, as long as inflation affects the borrower's income and the liquidation value of the home equally, it would not change the properties of the optimal mortgage in real terms. We also did not allow for contract renegotiations, because a possibility of renegotiation would lead to a suboptimal contract. In practice, lenders should be able to commit to the terms of a mortgage contract, or make renegotiation very costly for borrowers.

We assumed that the liquidation values do not depend on the interest rate. Even if house prices were sensitive to the interest rate, a documented significant delay[23] from time the foreclosure process is initiated till foreclosure sale would make the liquidation values much less sensitive to the interest rate compared to house prices. The independence of liquidation values from the interest rate was meant to capture this reality in our setting. We also assumed that the borrower's income process does not depend on the interest rate. This assumption is motivated by our view of the borrower as borrowing constrained household as well as the empirical evidence[24] indicating that most household income shocks are correlated only weakly with asset returns.

For the sake of tractability of our dynamic contracting problem, we had to assume risk-neutrality of the borrower with respect to luxury consumption. The properties of the optimal mortgage are determined

[22] See, for example, Campbell and Cocco (2003).

[23] Ambrose et al. (1997) report that borrowers whose delinquency continues to foreclosure spend an average of 13.8 total months in default before their property rights are terminated.

[24] See for example Campbell et al. (2000).

by the conflict of interest between the borrower and the lender and by the gains from trade based on the differences between the borrower's and the lender's discount factors. In particular, the adjustable features of the optimal mortgage are driven by the fact that the lender values his relationship with the borrower more when his discount rate is low. Therefore, we expect that a more general form of risk-aversion in our model would weaken but not completely eliminate the adjustable features of the optimal mortgage. However, solving the model with risk-aversion would require development of a completely new solution method.

There are a number of research directions one might pursue from here. In this paper we have considered time-homogeneous setting, in which agents are infinitely lived and the borrower's average income and the liquidation values of the home do not change over time. Relaxing this assumptions would allow us to study the effects of home appreciation trends and households' life-cycle income profiles on optimal mortgage design. Another avenue of research would be to extend our analysis to a general equilibrium framework and to study what effects the presence of private information in the mortgage origination market have on equilibrium home prices, and how this varies over the business cycle.

Appendix

A.1 Proofs of Lemmas and Propositions

Proof of Lemma 1

Consider any incentive compatible allocation $(\tau, I, C, \hat{Y})$. We prove the lemma by showing the existence of the new incentive-compatible allocation that that has the following properties:

(i) the borrower gets the same expected utility as under the old allocation (τ, I),

(ii) the borrower chooses to reveal the cash flows truthfully,

(iii) the borrower maintains zero savings,

(iv) the lender gets the same or greater expected profit as under the old allocation (τ, I).

Consider the candidate incentive compatible allocation (τ', I', C, Y) where

$$\begin{aligned} \tau'(Y, r) &= \tau(\hat{Y}(Y, r), r), \\ I'(Y, r) &= C(Y, r). \end{aligned}$$

We observe that the borrower's consumption and the termination time under the new allocation and the proposed borrower's response strategy, (C, Y), are the same as under the old allocation, so he earns the same expected utility, which establishes property (i). Also, by construction, the proposed response of the borrower to the allocation (τ', I') involves truth-telling and zero savings, which establishes properties (ii) and (iii).

Now we will show that (C, Y) is the borrower's incentive compatible strategy under the allocation (τ', I'). We note that the strategy (C, Y) yields the same utility to the borrower under the allocation (τ', I') as the incentive compatible strategy associated with the allocation (τ, I). Therefore, to show that (C, Y) is the borrower's incentive compatible strategy under the allocation (τ', I'), it is enough to show that if any alternative strategy (C', Y') is feasible under the allocation (τ', I'), then C' is also feasible under the old allocation (τ, I).

It follows that if C' is feasible under the new allocation, then the borrower has nonnegative savings if he reports $\hat{Y}(Y'(Y, r), r)$ and consumes C' under the old allocation, and thus C' is also feasible under the old allocation (τ, I). To see this we note that that the borrower's savings at any time $t \leq \tau(\hat{Y}(Y'(Y, r), r) =$

$\tau'(Y'(Y,r),r)$ under the old allocation (τ, I) and the borrower's strategy $(C', \hat{Y}(Y'(Y,r),r))$ are equal to

$$\underbrace{\int_0^t e^{\rho_t(t-s)}\left[dY_s - d\hat{Y}_s(Y'(Y,r),r) + dI_s(\hat{Y}(Y'(Y,r),r) - dC'_s(Y,r)\right]}_{\text{Savings under the old allocation, the borrower's strategy } (C',\hat{Y}(Y'(Y,r),r))\text{, and the realized } (Y,\, r)} =$$

$$\underbrace{\int_0^t e^{\rho_t(t-s)}\left[dY'_s(Y,r) - d\hat{Y}_s(Y'(Y,r),r) + dI_s(\hat{Y}(Y'(Y,r),r) - dC_s(Y'(Y,r),r)\right]}_{(\geq 0)\text{ Savings under the old allocation given the borrower's strategy } (C,\hat{Y}(Y'(Y,r),r))\text{, and the realized } (Y'(Y,r),r)}$$

$$\underbrace{+\int_0^t e^{\rho_t(t-s)}\left[dY_s - dY'_s(Y,r) + \underbrace{dC_s(Y'(Y,r),r)}_{=I'(Y'(Y,r),r)} - dC'_s(Y,r)\right]}_{(\geq 0)\text{ Savings under the new allocation, the borrower's strategy } (C,Y'(Y,r))\text{, and the realized } (Y,\, r)} \geq 0.$$

Finally, to complete the proof, we need to show that under the new allocation (τ', I') the lender gets the same or greater expected profit as under the allocation (τ, I). Note that under the new allocation the lender does savings for the borrower. As by assumption the lender's interest rate process is always greater or equal from the saving's interest rate available to the borrower (i.e., for all t, $r_t \geq \rho_t$), the lender's expected profit improves by

$$E_0\left[\int_0^\tau e^{-R_t}\left(r_t - \rho_t\right) S_t dt\right] \geq 0,$$

which shows (iv). $\square$

Proof of Proposition 3

Let b be a C^2 function (in a) that solves:

$$rb(a,r) = \mu + (\gamma a - \theta - \psi(a,r)\delta(r))b'(a_t,r_t) + \frac{1}{2}\sigma^2 b''(a,r) + \delta(r_t)\left(b(a_t + \psi(a,r), r^c) - b(a,r)\right) \qquad (51)$$

when a is in the interval $[A, a^1(r)]$, and $b'(a,r) = -1$ when $a > a^1(r)$, with boundary conditions $b(A,r) = L$ and

$$\begin{aligned}
\mu + \theta &= r_L b(a^1(r_L), r_L) + \gamma a^1(r_L) - \delta(r_L)\left[b(a^1(r_H), r_H) - b(a^1(r_L), r_L) + a^1(r_H) - a^1(r_L)\right], \\
\mu + \theta &= r_H b(a^1(r_H), r_H) + \gamma a^1(r_H) - \delta(r_H)\left[b(a^1(r_L), r_L) - b(a^1(r_H), r_H) + a^1(r_L) - a^1(r_H)\right],
\end{aligned}$$

where

$$\psi(a,r) = \begin{cases} \text{is a } C^1 \text{ (in } a\text{) solution to } b'(a,r) = b'(a+\psi, r^c) \text{ for all } (a,r) \\ \text{for which the solution is such that } \psi(a,r) > A - a \\ \\ \text{otherwise it is equal to } A - a \end{cases},$$

and where $r \in \{r_L, r_H\}$ and $r^c = \{r_L, r_H\} \setminus \{r\}$.

We start by showing that the function b is strictly concave in a over $[a, a^1(r)]$.

Lemma 4 *The function b is strictly concave over* $[a, a^1(r)]$.

Proof We will proceed in the series of steps below.

Step 1. Strict concavity in the neighborhood of the reflection barriers: Define the social surplus function as:

$$F(a,r) = a + b(a,r). \tag{52}$$

Then the social surplus function satisfies the following differential equation:

$$\begin{gathered} r(F(a,r) - a) = \mu + (\gamma a - \theta - \psi(a,r)\delta(r))(F'(a,r) - 1) + \frac{1}{2}\sigma^2 F''(a,r) \\ +\delta(r)\left(F(a+\psi(a,r), r^c) - F(a,r) + \psi(a,r)\right) \end{gathered}$$

which equals

$$\begin{gathered} rF(a,r) = -(\gamma - r)a + \mu + \theta + (\gamma a - \theta - \psi(a,r)\delta(r))F'(a,r) + \frac{1}{2}\sigma^2 F''(a,r) \\ +\delta(r)\left(F(a+\psi(a,r), r^c) - F(a,r)\right) \end{gathered} \tag{53}$$

with the boundary conditions:

$$\begin{aligned} F(A,r) &= A + L, \\ F'(a^1(r), r) &= 0, \\ F''(a^1(r), r) &= 0. \end{aligned}$$

Let's now focus on $a \in [\bar{a}(r), a^1(r)]$ when $\bar{a}(r) \geq A$ is the smallest a such that $b'(a,r) = b'(a+\psi, r^c)$ holds for all $a \in [\bar{a}(r), a^1(r)]$. This together with (52) implies that $F'(a,r) = F'(a+\psi, r^c)$ holds for all $a \in [\bar{a}(r), a^1(r)]$. Differentiating (53) with respect to $a \in [\bar{a}(r), a^1(r)]$ and using $F'(a,r) = F'(a+\psi, r^c)$ that holds for all $a \in [\bar{a}(r), a^1(r)]$ and the boundary conditions implies that

$$\frac{dF''(a^1(r)_-, r)}{da} = \frac{2(\gamma - r)}{\sigma^2} > 0.$$

Note that as $F''(a^1(r), r) = 0$ and as we have that $\frac{dF''(a^1(r)_-, r)}{da} > 0$ it implies that there exists $\varepsilon > 0$ such that $F''(a, r) < 0$ over the interval $(a^1(r) - \varepsilon, a^1(r))$. Also as $F'(a^1(r)) = 0$ and $F''(a, r) < 0$ over the interval $(a^1(r) - \varepsilon, a^1(r))$ it implies that $F'(a, r) > 0$ over the interval $(a^1(r) - \varepsilon, a^1(r))$.

Step 2. Strict concavity of $b(a, r)$ **over** $[\tilde{a}(r), a^1(r)]$, $\tilde{a}(r) \geq \bar{a}(r)$, **for which** $\psi(a, r) \leq 0$**:** Let's pick $r \in \{r_L, r_H\}$ and the smallest $\tilde{a}(r) \geq \bar{a}(r)$ such that $\psi(a, r) \leq 0$ holds over the interval $[\tilde{a}(r), a^1(r)]$. Such $\tilde{a}(r)$ must exists given that over $[\bar{a}(r), a^1(r)]$ we have that

$$\psi(a, r) = -\psi(a + \psi(a, r), r^c), \tag{54}$$

which follows from the definition of function ψ.

From (53) we have that

$$F''(a, r) = \frac{K(a, r) - (\gamma a - \theta) F'(a, r)}{2\sigma^2}, \tag{55}$$

where

$$K(a, r) = rF(a, r) + (\gamma - r)a + \psi(a, r)\delta(r)F'(a, r) - \delta(r)\left(F(a + \psi(a, r), r^c) - F(a, r)\right) - (\mu + \theta).$$

Now we note that $K(a^1(r), 0) = 0$ and that

$$K'(a, r) = rF'(a, r) + (\gamma - r) + \psi(a, r)\delta(r)F''(a, r) \tag{56}$$

which holds over $[A, a^1(r)]$.

Then (55) and (56), and the fact that $F''(a^1(r)_-, r) < 0$, imply that as long as $F'(a, r) > 0$ over $[\tilde{a}(r), a^1(r))$ we have that $F''(a, r) < 0$ in this interval. To see this note that (55) and (56) imply that we cannot have $F'(a, r) > 0$ over $[\tilde{a}(r), a^1(r))$ and at the same time $F''(a, r) = 0$ in this interval, as for the largest $\hat{a} \in [\tilde{a}(r), a^1(r))$ for which $F'' = 0$ we would have that $K(\hat{a}, r) < 0$ as K would be increasing on $[\hat{a}, a^1(r))$ and $K(a^1(r), 0) = 0$. But then given that $F'(\hat{a}, r) > 0$, $K(\hat{a}, r) < 0$, and $(\gamma\hat{a} - \theta) > 0$ (Assumption 1) we would have by (56) that $F''(\hat{a}, r) < 0$, which is a contradiction. So as long as $F'(a, r) > 0$ over $[\tilde{a}(r), a^1(r))$ we would have that $F''(a, r) < 0$ over $[\tilde{a}(r), a^1(r))$.

Now we remember that $F'(a, r) > 0$ in $[a^1(r) - \varepsilon, a^1(r))$. We want to show that $F'(a, r) > 0$ over the entire $[\tilde{a}(r), a^1(r))$, and thus by the above discussion that $F(a, r)$ is strictly concave over $[\tilde{a}(r), a^1(r))$. Now suppose by contradiction that $F' \leq 0$ for some $\tilde{a}(r) \leq a \leq a^1(r) - \varepsilon$, and let $\hat{a} = \sup\left\{a \leq a^1(r) - \varepsilon : F'(., r) \leq 0\right\}$. Then it follows that $F'(\hat{a}, r) = 0$, and that for all $a \in (\hat{a}, a^1(r))$ we have that $F' > 0$. But this implies that $F''(a, r) < 0$ for $a \in (\hat{a}, a^1(r))$. From the Fundamental Theorem of Calculus it follows that:

$$F'(a^1(r), r) = F'(\hat{a}, r) + \int_{\hat{a}}^{a^1(r)} F''(a, r)da,$$

which given that $F'(a^1(r), r) = 0$ implies that

$$F'(\hat{a}, r) = -\int_{\hat{a}(r)}^{a^1(r)} F''(a, r) da$$

As $F'' < 0$ for $a \in (\hat{a}, a^1(r))$ the above implies that $F'(\hat{a}, r) > 0$, which is a contradiction. Hence we have that $F' > 0$ for $a \in [\tilde{a}(r), a^1(r))$ and hence $F''(a, r) < 0$ for $a \in [\tilde{a}(r), a^1(r))$. Also if $\tilde{a}(r) > \bar{a}(r)$, and noting that F is C^2 (in a) and $F''(\tilde{a}(r), r) < 0$, there exists $\varepsilon > 0$ such that $F(a, r)$ is strictly concave and increasing over the interval $(\tilde{a}(r) - \varepsilon, \tilde{a}(r))$.

Step 3. Strict concavity of $b(a, r^c)$ over $[\tilde{a}(r^c), a^1(r^c))$, $\tilde{a}(r^c) = \tilde{a}(r) + \psi(\tilde{a}(r), r)$, where $\psi(a, r^c) \geq 0$: From (54) it follows that if $\psi(a, r) \leq 0$ holds over the interval $[\tilde{a}(r), a^1(r)]$, then we have that $\psi(a, r^c) \geq 0$ holds over the interval $[\tilde{a}(r^c), a^1(r^c)]$. Now note that for $a \in [\tilde{a}(r^c), a^1(r^c))$ we have that

$$F'(a, r^c) = F'(a + \psi(a, r^c), r) \tag{57}$$

Now differentiating (57) we obtain

$$F''(a, r^c) = F''(a + \psi(a, r^c), r)(1 + \psi'(a, r^c)). \tag{58}$$

We note that (54) implies that for $a \in [\tilde{a}(r^c), a^1(r^c))$ we have that $(a + \psi(a, r^c)) \in [\tilde{a}(r), a^1(r))$ and so $F''(a + \psi(a, r^c), r) < 0$. Now suppose that by contradiction $F''(a, r^c) \geq 0$ for some $a \in [\tilde{a}(r^c), a^1(r^c))$. As in the neighborhood of reflecting barriers we have that $F''(a, r^c) < 0$, and as the function F is continuous it would imply the existence of $\hat{a} \in [\tilde{a}(r^c), a^1(r^c))$ such that $F''(\hat{a}, r^c) = 0$. But then (58) and the fact that $F''(\hat{a}, r) < 0$ would imply that

$$\psi'(\hat{a}, r^c) = -1.$$

But then differentiating

$$\psi(a, r^c) = -\psi(a + \psi(a, r^c), r)$$

with respect to a and setting $a = \hat{a}$ yields

$$\psi'(\hat{a}, r^c) = -\psi'(\hat{a} + \psi(\hat{a}, r^c), r) \left[1 + \psi'(\hat{a}, r^c)\right],$$

which would imply that $-1 = 0$, which is a contradiction. Therefore we conclude that $F''(a, r^c) < 0$ and by (57) $F'(a, r^c) > 0$ for all $a \in [\tilde{a}(r^c), a^1(r^c))$. Also if $\tilde{a}(r^c) > \bar{a}(r^c)$, and given that F is C^2 (in a) and $F''(\tilde{a}(r^c), r^c) < 0$, there exists $\varepsilon > 0$ such that $F(a, r^c)$ is strictly concave and increasing over the interval $(\tilde{a}(r^c) - \varepsilon, \tilde{a}(r^c))$.

Step 4: Strict concavity of $F(a,r)$ over $[\bar{a}(r), a^1(r))$: If $\tilde{a}(r) = \bar{a}(r)$ it follows by Steps 1, 2, and 3. Suppose that $\tilde{a}(r) > \bar{a}(r)$. Then by Steps 1, 2, and 3 we know that the function $F(a,r)$ is strictly increasing and concave (in a) over the interval $(\tilde{a}(r) - \varepsilon, a^1(r))$. Then let's pick $r' \in \{r_L, r_H\}$ and the smallest $\tilde{a}'(r) \geq \bar{a}(r)$ such that $\psi(a, r'') \leq 0$ holds over the interval $[\tilde{a}'(r'), \tilde{a}(r')]$. Then applying the same reasoning as in Steps 3 and 4 we get that the functions $F(a,r')$ and $F(a,r'^c)$ are strictly increasing and concave over, respectively, $[\tilde{a}'(r'), a^1(r')]$ and $[\tilde{a}'(r'^c), a^1(r'^c)]$, where $\tilde{a}(r'^c) = \tilde{a}' + \psi(\tilde{a}', r')$. Applying this argument over and over again we obtain that the function $F(a,r)$ is strictly increasing and concave over the interval $[\bar{a}(r), a^1(r))$.

Step 5. Strict concavity of F over $[A, \max(\bar{a}(r), \bar{a}(r^c)]$: It follows from the definition of function F that $\min(\bar{a}(r), \bar{a}(r^c)) = A$ and then Steps 1, 2, 3, and 4 imply that for $\hat{r} \in \{r_L, r_H\}$ for which $\bar{a}(\hat{r}) = A$ we have that $F(a, \hat{r})$ is strictly concave and increasing over $[A, a^1(\hat{r})]$. From the definition of function F it follows that $\psi(a, \hat{r}^c) = A - a$ over $[A, \bar{a}(\hat{r}^c)]$. By steps 1, 2, 3, 4 we know that $F(a, \hat{r}^c)$ is strictly increasing and concave over $[\bar{a}(\hat{r}^c), a^1(\hat{r}^c)]$, and moreover because F is C^2 (in a) there exists $\varepsilon > 0$ such that $F(a, \hat{r}^c)$ is strictly increasing and concave over $(\bar{a}(\hat{r}^c) - \varepsilon, \bar{a}(\hat{r}^c)]$. But then because $\psi(a, \hat{r}^c) = A - a \leq 0$ over $[A, \bar{a}(\hat{r}^c)]$, applying the same reasoning as in Step 2 yields us that $F(a, \hat{r}^c)$ is strictly increasing and concave over $[A, \bar{a}(\hat{r}^c)]$ and so is strictly increasing and concave over $[A, a^1(\hat{r}^c)]$.

Steps 1, 2, 3, 4, and 5 imply that F is strictly increasing and concave over $[A, a^1(r))$ and as $b''(a,r) = F''(a,r)$ for any $a \geq A$ and $r \in \{r_L, r_H\}$ we have that $b(a,r)$ is strictly concave (in a) over $[A, a^1(r))$. ■

Now for any incentive compatible allocation (τ, I, C, Y) we define:

$$G_t = \int_0^t e^{-R_s}(dY_s - dI_s) + e^{-R_t} b(a_t, r_t), \tag{59}$$

where a_t evolves according to (6). We note that the process G is such that G_t is $\mathcal{F}_t-$measurable.

We remember that under an arbitrary incentive compatible allocation, (τ, I, C, Y), a_t evolves as

$$da_t(r_t) = (\gamma a_t - \theta - \psi_t \delta(r_t))\, dt - dI_t + \beta_t dZ_t + \psi_t dN_t.$$

where $\beta_t \geq \sigma$ m-a.s. for any $0 \leq t \leq \tau$. From Ito's lemma we get that

$$\begin{aligned} db(a_t, r_t) = &\left[(\gamma a_t - \theta - \psi_t \delta(r_t)) b'(a_t, r_t) + \frac{1}{2}\beta_t^2 b''(a_t, r_t)\right] dt - b'(a_t, r_t) dI_t \\ &+ \beta_t b'(a_t, r_t) dZ_t + \left[b(a_t + \psi_t, r_t^c) - b(a_t, r_t)\right] dN_t. \end{aligned}$$

Then combining the above with (59) yields

$$
\begin{aligned}
e^{R_t} dG_t &= \left[\mu + (\gamma a_t - \theta - \psi_t \delta(r_t)) b'(a_t, r_t) + \frac{1}{2}\beta_t^2 b''(a_t, r_t) - r_t b'(a_t, r_t)\right] dt \\
&\quad -(1 + b'(a_t, r_t)) dI_t + (\sigma + \beta_t b'(a_t, r_t))\, dZ_t + \left[b(a_t + \psi_t, r_t^c) - b(a_t, r_t)\right] dN_t
\end{aligned}
$$

Combining the above with (51) yields

$$
\begin{aligned}
e^{R_t} dG_t &\leq \left[\frac{1}{2}\left(\beta_t^2 - \sigma^2\right) b''(a_t, r_t) + \delta(r_t) b'(a_t, r_t) \left[\psi(a_t, r_t) - \psi_t\right]\right] dt - (1 + b'(a_t, r_t)) dI_t \\
&\quad + (\sigma + \beta_t b'(a_t, r_t))\, dZ_t + \left[b(a_t + \psi_t, r_t^c) - b(a_t + \psi(a_t, r_t), r_t^c)\right] dN_t,
\end{aligned}
$$

with equality whenever $a \in [A, a^1(r_t)]$. From the above we have that for any $0 \leq t < \tau$:

$$
\begin{aligned}
e^{R_t} dG_t &\leq \underbrace{\left[\frac{1}{2}\left(\beta_t^2 - \sigma^2\right) b''(a_t, r_t)\right]}_{\leq 0} dt \underbrace{-(1 + b'(a_t, r_t)) dI_t}_{\leq 0} \\
&\quad + \underbrace{\delta(r_t) \left(\left[b(a_t + \psi_t, r_t^c) - \psi_t b'(a_t, r_t)\right] - \left[b(a_t + \psi(a_t, r_t), r_t^c) - \psi(a_t, r_t) b'(a_t, r_t)\right]\right)}_{\leq 0} dt \\
&\quad + (\sigma + \beta_t b'(a_t, r_t))\, dZ_t + \left[b(a_t + \psi_t, r_t^c) - b(a_t + \psi(a_t, r_t), r_t^c)\right] dM_t,
\end{aligned}
\tag{60}
$$

with equality whenever $a \in [A, a^1(r_t)]$. The first component of the RHS of the above inequality is less or equal to zero because the function b is concave (Lemma 4) and $\beta_t \geq \sigma$ for any $t \leq \tau$. The second component is less or equal to zero because $b' \geq -1$ and $dI_t \geq 0$. The third component is less or equal to zero because, by definition, the function ψ is a solution to

$$
\max_{\psi \geq A - a} \left[b(a + \psi, r^c) - \psi b'(a, r)\right].
$$

The condition (60) implies that the process G is an $\mathcal{F}_t-$supermartingale up to time $t = \tau$, where we recall that Z and M are martingales. It will be an $\mathcal{F}_t-$martingale if and only if, for $t > 0$, $a_t \leq a^1(r_t)$, $\beta_t = \sigma$ m-a.s., $\psi_t = \psi(a_t, r_t)$, and I_t is increasing only when $a_t \geq a^1(r_t)$.

We now evaluate the lender's expected utility for an arbitrary incentive compatible allocation (τ, I, C, Y), which equals

$$
E\left[\int_0^\tau e^{-R_s}(dY_s - dI_s) + e^{-R_\tau} L\right].
$$

We note that $b(a_\tau, r_\tau) = L$ as, from the definition of a, $a_\tau = A$. Using this, and the definition of process G, we have that under any arbitrary incentive compatible allocation (τ, I, C, Y) and any $t \in [0, \infty)$:

$$E\left[\int_0^\tau e^{-R_s}(dY_s - dI_s) + e^{-R_\tau}L\right] =$$

$$E\left[G_{t\wedge\tau}\right] + E\left[1_{t\le\tau}\left(\int_t^\tau e^{-R_s}(dY_s - dI_s) + e^{-R_\tau}L - e^{-R_t}b(a_t, r_t)\right)\right] \le$$

$$b(a_0, r_0) + E\left[1_{t\le\tau}\left(\int_t^\tau e^{-R_s}(dY_s - dI_s) + e^{-R_\tau}L - e^{-R_t}b(a_t, r_t)\right)\right] =$$

$$b(a_0, r_0) + e^{-R_t}E\left[1_{t\le\tau}\left(E\left[\int_t^\tau e^{R_t - R_s}(dY_s - dI_s) + e^{R_t - R_\tau}L\,|\mathcal{F}_t\right] - b(a_t, r_t)\right)\right], \tag{61}$$

where, the inequality follows from the fact that $G_{t\wedge\tau}$ is supermartingale and $G_0 = b(a_0, r_0)$. We note that in the above

$$E\left[\int_t^\tau e^{R_t - R_s}(dY_s - dI_s) + e^{R_t - R_\tau}L\,|\mathcal{F}_t\right] < \frac{\mu}{r_L} + \frac{\theta}{\gamma} - a_t,$$

as the RHS of the above inequality is the upper bound on the lender's expected profit under the first-best (public information) allocation. Using the above inequality in (61) we have that

$$E\left[\int_0^\tau e^{-R_s}(dY_s - dI_s) + e^{-R_\tau}L\right] \le b(a_0, r_0) + e^{-R_t}E\left[1_{t\le\tau}\left(\frac{\mu}{r_L} + \frac{\theta}{\gamma} - a_t - b(a_t, r_t)\right)\right].$$

Using $b'(a, r) \ge -1$, we have that, for any $a \ge A$, $-a - b(a, r) \le -A - L$. Applying this to the above inequality yields

$$E\left[\int_0^\tau e^{-R_s}(dY_s - dI_s) + e^{-R_\tau}L\right] \le b(a_0, r_0) + e^{-R_t}E\left[1_{t\le\tau}\left(\frac{\mu}{r_L} + \frac{\theta}{\gamma} - A - L\right)\right].$$

Taking $t \to \infty$ yields

$$E\left[\int_0^\tau e^{-R_s}(dY_s - dI_s) + e^{-R_\tau}L\right] \le b(a_0, r_0).$$

Let (τ^*, I^*, C^*, Y) be an allocation satisfying the conditions of the proposition. We remember that this allocation is incentive compatible as it is feasible and $\beta_t = \sigma \ge \sigma$ for any $t \le \tau$. Also under this allocation the process G_t is a martingale until time τ (note that $b'(a,r)$ is bounded). So we have that

$$E\left[\int_0^{\tau^*} e^{-R_s}(dY_s - dI_s^*) + e^{-R_{\tau^*}}L\right] =$$

$$b(a_0,r_0) + e^{-R_t}E\left[1_{t\leq\tau^*}\left(E\left[\int_t^{\tau^*} e^{R_t-R_s}(dY_s - dI_s^*) + e^{R_t-R_{\tau^*}}L\,|\mathcal{F}_t\right] - b(a_t,r_t)\right)\right].$$

Taking $t\to\infty$ and using

$$\lim_{t\to\infty} e^{-R_t}E\left[1_{t\leq\tau^*}\left(E\left[\int_t^{\tau^*} e^{R_t-R_s}(dY_s - dI_s^*) + e^{R_t-R_{\tau^*}}L\,|\mathcal{F}_t\right] - b(a_t,r_t)\right)\right] = 0,$$

yields

$$E\left[\int_0^{\tau^*} e^{-R_s}(dY_s - dI_s^*) + e^{-R_{\tau^*}}L\right] = b(a_0,r_0). \quad \square$$

Proof of Proposition 4

Let $(C,\hat{Y})$ be any borrower's feasible strategy given the allocation (τ, I). The borrower's private saving's account balance, S, under the strategy $(C,\hat{Y})$ and the allocation (τ, I) grows, for $t\in[0,\tau]$, according to

$$dS_t = \rho_t S_t dt + (dY_t - d\hat{Y}_t) + dI_t - dC_t, \tag{62}$$

where we remember that $\rho_t \leq r_t$. Define the process $\hat{V}$ as

$$\hat{V}_t = \int_0^t e^{-\gamma s}dC_s + \int_0^t e^{-\gamma s}\theta ds + e^{-\gamma t}(S_t + a_t),$$

From the above it follows that

$$e^{\gamma t}d\hat{V}_t = dC_t + \theta dt + dS_t - \gamma S_t dt + da_t - \gamma a_t dt$$

Using (16) and (62) yields

$$\begin{aligned} e^{\gamma t}d\hat{V}_t &= (\rho_t - \gamma)S_t dt + (dY_t - \mu dt)dt + \psi_t dM_t = \\ &\quad (\rho_t - \gamma)S_t dt + \sigma dZ_t + \psi_t dM_t. \end{aligned} \tag{63}$$

Noting that $e^{\gamma t} \geq 1$ for any $t \geq 0$, we have that

$$d\hat{V}_t \leq (\rho_t - \gamma) S_t dt + \sigma dZ_t + \psi_t dM_t.$$

Recall that Z and M are martingales, $\rho_t < \gamma$, and that the process S is nonnegative. So it follows from the above that the process $\hat{V}$ is supermartingale up to time τ (note that a is bounded from below). Using this and the fact that by definition $a_\tau = A$, we have that for any feasible strategy of the borrower,

$$a_0 = \hat{V}_0 \geq E\left[\hat{V}_\tau\right] = E\left[\int_0^\tau e^{-\gamma s} dC_s + \int_0^\tau e^{-\gamma s}\theta ds + e^{-\gamma\tau}(S_\tau + A)\right], \tag{64}$$

The right-hand-side of (64) represents the expected utility for the borrower under any feasible $\left(C, \hat{Y}, S\right)$. This utility is bounded by a_0. If the borrower maintains zero savings, $S_t = 0$, reports cash flows truthfully, $d\hat{Y}_t = dY_t$, then $\hat{V}$ is a martingale up to time τ, which means that (64) holds with equality and the borrower's expected utility is a_0. Thus, this is the optimal strategy for the borrower. $\square$

Proof of Proposition 5

Define $\tilde{a}_t$ as follows:

$$\begin{aligned} \tilde{a}_t &= A + C_t^L(r_t) - B_t && (65) \\ &= p_t + a^1(r_t) - B_t. && (66) \end{aligned}$$

Under the candidate mortgage contract the debt balance evolves according to

$$dB_t = \left(\bar{r}_t^p B_t + (\bar{r}_t - \bar{r}_t^p)(B_t - p_t)^+\right) dt - d\hat{Y}_t + dI_t, \tag{67}$$

when $B_t \leq C_t^L$, where I_t represents cumulative withdrawal of money by the borrower. In addition,

$$\begin{aligned} dC_t^L &= dp_t + da^1(r_t) \\ &= \psi(p_t + a^1(r_t) - B_t, r_t) dN_t && (68) \end{aligned}$$

Using (17)-(19), (66)-(68), for $B_t \geq p_t$ we can write

$$\begin{aligned}
d\tilde{a}_t &= dC_t^L(r_t) - dB_t \\
&= \psi(p_t + a^1(r_t) - B_t, r_t)dN_t - \left(\bar{r}_t^p B_t + \left(\bar{r}_t - \bar{r}_t^p\right)(B_t - p_t)\right)dt + d\hat{Y}_t - dI_t \\
&= -\left(\bar{r}_t^p p_t + \bar{r}_t\left(B_t - p_t\right) - \delta\psi(p_t + a^1(r_t) - B_t, r_t)\right)dt + d\hat{Y}_t - dI_t \\
&\quad + \psi(p_t + a^1(r_t) - B_t, r_t)dM_t \\
&= -\left(\theta + \mu - \gamma a^1(r_t) + \gamma\left(B_t - p_t\right)\right)dt + d\hat{Y}_t - dI_t + \psi(p_t + a^1(r_t) - B_t, r_t)dM_t \\
&= \gamma\tilde{a}_t dt - \mu dt - \theta dt + d\hat{Y}_t - dI_t + \psi(\tilde{a}_t, r_t)dM_t
\end{aligned} \tag{69}$$

The borrower's savings evolve according to

$$dS_t = \rho_t S_t dt + dI_t + \left(dY_t - d\hat{Y}_t\right) - dC_t. \tag{70}$$

Consider

$$\hat{V}_t = \int_0^t e^{-\gamma s}\left(\theta dt + dC_s\right) + e^{-\gamma t}\left(\Omega_t + S_t\right)$$

where

$$\Omega_t = \begin{cases} a^1\left(r_t\right) + \left(p_t - B_t\right),\ if\ B_t < p_t \\ \tilde{a}_t,\ if\ B_t \geq p_t \end{cases}. \tag{71}$$

We will show that for any feasible strategy $\left(C, \hat{Y}, S\right)$ of the borrower, $\hat{V}_t$ is a supermartingale. Note that

$$d\Omega_t = \begin{cases} \underbrace{\left[a^1(r_t^c) - a^1(r_t)\right]}_{\psi(a^1(r_t), r_t)} dN_t - dB_t,\ if\ B_t < p_t \\ d\tilde{a}_t,\ if\ B_t \geq p_t \end{cases}. \tag{72}$$

Using (70),

$$\begin{aligned}
e^{\gamma t} d\hat{V}_t &= \theta dt + dC_t + dS_t - \gamma S_t dt + d\Omega_t - \gamma\Omega_t dt \\
&= \theta dt - \left(\gamma - \rho_t\right)S_t dt + dI_t + \left(dY_t - d\hat{Y}_t\right) + d\Omega_t - \gamma\Omega_t dt.
\end{aligned}$$

First, we consider the case with $B_t \geq p_t$. Using (1), (69), and (71)-(72),

$$\begin{aligned}
e^{\gamma t} d\hat{V}_t &= \theta dt - \left(\gamma - \rho_t\right)S_t dt + dI_t + \left(dY_t - d\hat{Y}_t\right) + d\tilde{a}_t - \gamma\tilde{a}_t dt \\
&= \left(\rho_t - \gamma\right)S_t dt + \sigma dZ_t + \psi(\tilde{a}_t, r_t)dM_t.
\end{aligned} \tag{73}$$

Now, let $B_t < p_t$. Using (1), (18), (67)-(72) yields

$$\begin{aligned}
e^{\gamma t} d\hat{V}_t &= \theta dt - (\gamma - \rho_t) S_t dt + dI_t + \left(dY_t - d\hat{Y}_t\right) + d\Omega_t - \gamma \Omega_t dt \\
&= \theta dt - (\gamma - \rho_t) S_t dt + dI_t + \left(dY_t - d\hat{Y}_t\right) \\
&\quad + \psi(a^1(r_t), r_t) dN_t - dB_t - \gamma \left(a^1(r_t) + (p_t - B_t)\right) dt \\
&= -\left(\bar{r}_t^p B_t + \gamma (p_t - B_t) + \gamma a^1(r_t) - \theta - \mu + (\gamma - \rho_t) S_t\right) dt \\
&\quad + \psi(a^1(r_t), r_t) dN_t + \sigma dZ_t \\
&= -\left(\bar{r}_t^p B_t + \gamma (p_t - B_t) - \bar{r}_t^p p_t + \delta \psi(a^1(r_t), r_t) + (\gamma - \rho_t) S_t\right) dt \\
&\quad + \psi(a^1(r_t), r_t) dN_t + \sigma dZ_t \\
&= -(\gamma - \bar{r}_t^p)(p_t - B_t) dt - (\gamma - \rho_t) S_t dt + \psi(a^1(r_t), r_t) dM_t + \sigma dZ_t \qquad (74)
\end{aligned}$$

Recall that Z and M are martingales, $\bar{r}_t^p \leq \gamma$, $\rho_t < \gamma$. Thus, it follows from (73), (74), and the fact that $\tilde{a}_t$ is bounded from below, that for any feasible strategy $\left(C, \hat{Y}, S\right)$ of the borrower $\hat{V}_t$ is a supermartingale until default time $\tau\left(C, \hat{Y}, S\right) = \inf\left\{t : B_t = C_t^L\right\}$. Since $\Omega_\tau = A$,

$$\begin{aligned}
A + C_0^L(r_0) - B_0 &= \tilde{a}_0 = \hat{V}_0 \geq E\left[\hat{V}_{\tau\left(C,\hat{Y},S\right)}\right] \\
&= E\left[\int_0^{\tau\left(C,\hat{Y},S\right)} e^{-\gamma s}(\theta dt + dC_s) + e^{-\gamma\tau\left(C,\hat{Y},S\right)}\left(A + S_{\tau\left(C,\hat{Y},S\right)}\right)\right], \qquad (75)
\end{aligned}$$

where B_0 is the time-zero draw on the credit line.

The right-hand-side of (75) represents the expected utility for the borrower under strategy $\left(C, \hat{Y}, S\right)$, given the terms of the mortgage. This utility is bounded by $A + C_0^L(r_0) - B_0$. If the borrower maintains zero savings, $S_t = 0$, reports cash flows truthfully, $d\hat{Y}_t = dY_t$, and consumes all excess cash flows once the balance on the credit line reaches $p_t(r_t)$, so that $B_t \geq p_t$ and $C_t = I_t^* = \max(0, p_t - B_t) = \max(0, \tilde{a}_t - a_t^1)$, then $\hat{V}_t$ is a martingale, which means that (75) holds with equality and the borrower's expected utility is $A + C_0^L(r_0) - B_0$. Thus, this is the optimal strategy for the borrower.

Reproducing the above argument for the borrower's optimal strategy, $(C, \hat{Y}, S) = (I^*, Y, 0)$, and the process $\hat{V}_{t'}$, $t' \leq \tau(I^*, Y, 0)$, defined in (82) yields that, for any $0 \leq t \leq \tau(C, \hat{Y}, 0)$, $\tilde{a}_t$ is equal to the borrower's continuation utility under the proposed mortgage contract with the initial expected utility for the borrower given by $a_0 = A + C_0^L(r_0) - B_0$, which establishes (21).

Under the proposed mortgage contract and the borrower's optimal strategy $(C, \hat{Y}, S) = (I^*, Y, 0)$, the lender's expected utility equals

$$E\left[\int_0^{\tau(I^*,Y,0)} e^{-R_t}(dY_t - dI_t^*) + e^{-R_{\tau(I^*,Y,0)}\tau(I^*,Y,0)} L \,|\mathcal{F}_0\right],$$

where

$$\tau(I^*, Y, 0) = \inf\left\{t : B_t = C_t^L\right\} = \inf\left\{t : \tilde{a}_t = A\right\} = \tau^*(Y),$$

as the borrower's continuation utility, $\tilde{a}$, evolve according to the equation (69), e.g. as in the optimal allocation. Therefore, we conclude that the proposed mortgage contract implements the optimal allocation. $\square$

Proof of Proposition 6

Consider the candidate mortgage contract. Under this contract the borrower's balance on the credit line evolves according to

$$dB_t = \bar{r}_t(B_t, r_t)B_t dt + x_t(r_t)dt - (d\hat{Y}_t - dI_t) + BA(B_t, r_t)dN_t, \tag{76}$$

when $B_t \leq C_t^L(r_t)$, while the borrower's savings evolve according to

$$dS_t = \rho_t S_t dt + dI_t + \left(dY_t - d\hat{Y}_t\right) - dC_t, \tag{77}$$

where I_t represents cumulative withdrawal of money from the credit line by the borrower.

Let $(C, \hat{Y}, S)$ be any borrower's feasible strategy under the proposed mortgage contract. For any feasible borrower's strategy $(C, \hat{Y}, S)$ define a process $\hat{V}$ as

$$\hat{V}_t = \int_0^t e^{-\gamma s}(dC_s + \theta ds) + e^{-\gamma t}\left(\tilde{a}_t + S_t\right), \tag{78}$$

where

$$\tilde{a}_t = a^1(r_t) - B_t \tag{79}$$

It follows from (76), (79), and (22)-(25) that $\tilde{a}$ evolves as

$$\begin{aligned}
d\tilde{a}_t &= \left[a^1(r_t^c) - a^1(r_t)\right] dN_t - dB_t \\
&= \left[a^1(r_t^c) - a^1(r_t)\right] dN_t - \bar{r}_t(B_t, r_t)B_t dt - x_t(B_t, r_t)dt - BA(B_t, r_t)dN_t + d\hat{Y}_t - dI_t \\
&= \left[a^1(r_t^c) - a^1(r_t)\right] dN_t - \left[\gamma\left(a^1(r_t) - \tilde{a}_t\right) - \delta(r_t)\left[\psi(a^1(r_t), r_t) - \psi(\tilde{a}_t, r_t)\right]\right] dt \\
&\quad - \left[\theta + \mu - \gamma a^1(r_t) + \delta(r_t)\psi(a^1(r_t), r_t)\right] dt - \left[-\psi(\tilde{a}_t, r_t) + (a^1(r_t^c) - a^1(r_t))\right] dN_t \\
&\quad + d\hat{Y}_t - dI_t \\
&= \left(\gamma\tilde{a}_t - \theta - \delta(r_t)\psi(\tilde{a}_t, r_t)\right) dt + (d\hat{Y}_t - \mu dt) - dI_t + \psi(\tilde{a}_t, r_t)dN_t
\end{aligned} \tag{80}$$

Using (1), (77), (78), (80) yields

$$\begin{aligned} e^{\gamma t} d\hat{V}_t &= dC_t + \theta dt + d\tilde{a}_t + dS_t - \gamma \tilde{a}_t dt - \gamma S_t dt \\ &= \sigma dZ_t + \psi(\tilde{a}_t, r_t) dM_t + (\rho_t - \gamma) S_t dt \end{aligned}$$

Recall that Z and M are martingales, $\rho_t < \gamma$, and that the process S is nonnegative. So it follows from the above that the process $\hat{V}$ is a supermartingale up to time $\tau\left(C, \hat{Y}, S\right) = \inf\left\{t : B_t = C_t^L\right\}$ (note that $\tilde{a}$ is bounded from below). Using this and the fact that by definition $\tilde{a}_\tau = A$, we have that for any feasible strategy of the borrower, $(C, \hat{Y}, S)$,

$$\begin{aligned} A + CL_0(r_0) - B_0 = a_0 = \tilde{a}_0 = \hat{V}_0 &\geq E\left[\hat{V}_{\tau\left(C,\hat{Y},S\right)}\right] \\ &= E\left[\int_0^{\tau\left(C,\hat{Y},S\right)} e^{-\gamma s}(dC_s + \theta ds) + e^{-\gamma\tau\left(C,\hat{Y},S\right)}\left(S_{\tau\left(C,\hat{Y},S\right)} + A\right)\right] \end{aligned} \tag{81}$$

The right-hand-side of (81) represents the expected utility for the borrower under any feasible strategy $\left(C, \hat{Y}, S\right)$, given the terms of the mortgage. This utility is bounded by $A + C_0^L(r_0) - B_0$, where B_0 is the initial draw on the credit line. If the borrower maintains zero savings, $S_t = 0$, reports cash flows truthfully, $d\hat{Y}_t = dY_t$, and consumes all excess cash flows once the balance on the credit line reaches 0, so that $C = I = I^* = \max(0, -B_t) = \max(0, \tilde{a}_t - a^1(r_t))$, then $\hat{V}$ is a martingale, which means that (81) holds with equality and the borrower's expected utility is $A + C_0^L(r_0) - B_0$. Thus, this is the optimal strategy for the borrower.

Reproducing the above argument for the borrower's optimal strategy, $(C, \hat{Y}, S) = (I^*, Y, 0)$, and the process $\hat{V}_{t'}$, $t' \leq \tau(I^*, Y, 0)$, defined as

$$\hat{V}_{t',t} = \int_{t'}^{t} e^{-\gamma(s-t')}(dC_s + \theta ds) + e^{-\gamma(t-t')}\tilde{a}_t, \quad t \geq t', \tag{82}$$

yields that, for any $0 \leq t \leq \tau(I^*, Y, 0)$, $\tilde{a}_t$ is equal to the borrower's continuation utility under the proposed mortgage contract with the initial expected utility for the borrower given by $a_0 = A + C_0^L(r_0) - B_0$, which establishes (26).

Under the proposed mortgage contract and the borrower's optimal strategy, the lender's expected utility equals

$$E\left[\int_0^{\tau(I^*,Y,0)} e^{-R_t}(dY_t - dI_t^*) + e^{-R_{\tau(I^*,Y,0)}\tau(I^*,Y,0)} L \,|\mathcal{F}_0\right],$$

where

$$\tau(I^*, Y, 0) = \inf\left\{t : B_t = C_t^L\right\} = \inf\left\{t : \tilde{a}_t = A\right\} = \tau^*(Y),$$

as the borrower's continuation utility, $\tilde{a}$, evolve according to the equation (80), e.g. as in the optimal allocation. Therefore, we conclude that the proposed mortgage contract implements the optimal allocation. $\square$

Proof of Proposition 7

Define $\tilde{a}_t$ as follows:

$$\begin{aligned} \tilde{a}_t &= A + C_t^L(r_t) - B_t && (83) \\ &= p_t + a^1(r_t) - B_t. && (84) \end{aligned}$$

Under the candidate mortgage contract the balance on the HELOC evolves according to

$$dB_t = \left(\bar{r}_t^p B_t + (\bar{r}_t - \bar{r}_t^p)\,(B_t - p_t)^+\right) dt + x_t(r_t)dt + BA^-(B_t - p_t)dN_t - d\hat{Y}_t + dI_t \quad (85)$$

when $B_t \leq C_t^L$, where I_t represents cumulative withdrawal of money from the credit line by the borrower. In addition,

$$\begin{aligned} dC_t^L &= dp_t + da^1(r_t) \\ &= \left(\psi(a^1(r_t) - (B_t - p_t, r_t)I_{r_{t\_}=r_L} + da^1(r_t)I_{r_{t\_}=r_H}\right) dN_t && (86) \end{aligned}$$

Using (84)-(86) and (27)-(32), for $B_t \geq p_t$, we can write

$$\begin{aligned} d\tilde{a}_t &= dC_t^L(r_t) - dB_t \\ &= \left(\psi(p_t + a^1(r_t) - B_t, r_t)I_{r_{t\_}=r_L} dN_t + da^1(r_t)I_{r_{t\_}=r_H}\right) dN_t - \left(\bar{r}_t^p B_t + (\bar{r}_t - \bar{r}_t^p)\,(B_t - p_t)\right) dt \\ &\quad -x_t(r_t) - BA^-(B_t - p_t)dN_t + d\hat{Y}_t - dI_t \\ &= \left(\psi(p_t + a^1(r_t) - B_t, r_t)I_{r_{t\_}=r_L} dN_t + da^1(r_t)I_{r_{t\_}=r_H}\right) dN_t - \left(\bar{r}_t^p B_t + (\bar{r}_t - \bar{r}_t^p)\,(B_t - p_t)\right) dt \\ &\quad -x_t(r_t) - (-\psi(p_t + a^1(r_t) - B_t, r_t)I_{r_{t\_}=r_H} dN_t + da^1(r_t)I_{r_{t\_}=r_H})dN_t + d\hat{Y}_t - dI_t \\ &= -\left(\theta + \mu - \gamma a^1(r_t) + \gamma\,(B_t - p_t)\right) dt + d\hat{Y}_t - dI_t + \psi(p_t + a^1(r_t) - B_t, r_t)dM_t \\ &= \gamma\tilde{a}_t dt - \mu dt - \theta dt + d\hat{Y}_t - dI_t + \psi(\tilde{a}_t, r_t)dM_t && (87) \end{aligned}$$

The borrower's savings evolve according to

$$dS_t = \rho_t S_t dt + dI_t + \left(dY_t - d\hat{Y}_t\right) - dC_t. \quad (88)$$

Consider

$$\hat{V}_t = \int_0^t e^{-\gamma s}\,(\theta dt + dC_s) + e^{-\gamma t}\,(\Omega_t + S_t)$$

where

$$\Omega_t = \begin{cases} a^1(r_t) + (p_t - B_t), \ if \ B_t < p_t \\ \tilde{a}_t, \ if \ B_t \geq p_t \end{cases}. \tag{89}$$

We will show that for any feasible strategy $\left(C, \hat{Y}, S\right)$ of the borrower, $\hat{V}_t$ is a supermartingale. Note that

$$d\Omega_t = \begin{cases} \underbrace{[a^1(r_t^c) - a^1(r_t)]}_{\psi(a^1(r_t), r_t)} dN_t - dB_t, \ if \ B_t < p_t \\ d\tilde{a}_t, \ if \ B_t \geq p_t \end{cases}. \tag{90}$$

Using (88),

$$\begin{aligned} e^{\gamma t} d\hat{V}_t &= \theta dt + dC_t + dS_t - \gamma S_t dt + d\Omega_t - \gamma \Omega_t dt \\ &= \theta dt - (\gamma - \rho_t) S_t dt + dI_t + \left(dY_t - d\hat{Y}_t\right) + d\Omega_t - \gamma \Omega_t dt. \end{aligned}$$

First, we consider the case with $B_t \geq p_t$. Using (1), (87), (89)-(90),

$$\begin{aligned} e^{\gamma t} d\hat{V}_t &= \theta dt - (\gamma - \rho_t) S_t dt + dI_t + \left(dY_t - d\hat{Y}_t\right) + d\tilde{a}_t - \gamma \tilde{a}_t dt \\ &= (\rho_t - \gamma) S_t dt + \sigma dZ_t + \psi(\tilde{a}_t, r_t) dM_t. \end{aligned} \tag{91}$$

Now, let $B_t < p_t$. Using (1), (27)-(31), (85)-(90) yields

$$\begin{aligned} e^{\gamma t} d\hat{V}_t &= \theta dt - (\gamma - \rho_t) S_t dt + dI_t + \left(dY_t - d\hat{Y}_t\right) \\ &\quad + \psi(a^1(r_t), r_t) dN_t - dB_t - \gamma \left(a^1(r_t) + (p_t - B_t)\right) dt \\ &= \theta dt - (\gamma - \rho_t) S_t dt + dI_t + \left(dY_t - d\hat{Y}_t\right) \\ &\quad + \psi(a^1(r_t), r_t) dN_t \\ &\quad - \left[\left(\bar{r}_t^p B_t + (\bar{r}_t - \bar{r}_t^p)(B_t - p_t)^+\right) dt + x_t(r_t) dt + BA^-(B_t - p_t) dN_t - d\hat{Y}_t + dI_t\right] \\ &\quad - \gamma \left(a^1(r_t) + (p_t - B_t)\right) dt \\ &= \theta dt - (\gamma - \rho_t) S_t dt + dY_t \\ &\quad + \psi(a^1(r_t), r_t) dN_t - [\theta + \mu - \gamma a^1(r_t) + \delta(r_t) \psi(a^1(r_t), r_t)] dt \\ &\quad - \gamma \left(a^1(r_t) + (p_t - B_t)\right) dt \\ &= -\gamma (p_t - B_t) dt - (\gamma - \rho_t) S_t dt + \psi(a^1(r_t), r_t) dM_t + \sigma dZ_t \end{aligned} \tag{92}$$

Recall that Z and M are martingales and that $\rho_t < \gamma$. Thus, it follows from (91), (92), and the fact that $\tilde{a}_t$ is bounded from below, that for any feasible strategy $\left(C, \hat{Y}, S\right)$ of the borrower $\hat{V}_t$ is a supermartingale until default time $\tau\left(C,\hat{Y},S\right)=\inf\left\{t:B_t=C_t^L\right\}$. Since $\Omega_\tau=A$,

$$
\begin{aligned}
A+C_0^L(r_0)-B_0 &= \tilde{a}_0+S_0=\hat{V}_0\geq E\left[\hat{V}_{\tau\left(C,\hat{Y},S\right)}\right] \\
&= E\left[\int_0^{\tau\left(C,\hat{Y},S\right)} e^{-\gamma s}\left(\theta dt+dC_s\right)+e^{-\gamma\tau\left(\hat{Y},C,S\right)}\left(A+S_{\tau\left(C,\hat{Y},S\right)}\right)\right], \qquad (93)
\end{aligned}
$$

where B_0 is the time-zero draw on the credit line.

The right-hand-side of (93) represents the expected utility for the borrower under strategy $\left(C,\hat{Y},S\right)$, given the terms of the mortgage. This utility is bounded by $A+C_0^L(r_0)-B_0'+S_0$. If the borrower maintains zero savings, $S_t=0$, reports cash flows truthfully, $d\hat{Y}_t=dY_t$, and consumes all excess cash flows once the balance on the credit line reaches $p_t(r_t)$, so that $B_t\geq p_t$ and $C_t=I_t^*=\max(0,p_t-B_t)=\max(0,\tilde{a}_t-a_t^1)$, then $\hat{V}_t$ is a martingale, which means that (75) holds with equality and the borrower's expected utility is $A+C_0^L(r_0)-B_0$. Thus, this is the optimal strategy for the borrower.

Reproducing the above argument for the borrower's optimal strategy, $(C,\hat{Y},S)=(I^*,Y,0)$, and the process $\hat{V}_{t'}$, $t'\leq\tau(I^*,Y,0)$, defined in (82) yields that, for any $0\leq t\leq\tau(C,\hat{Y},0)$, $\tilde{a}_t$ is equal to the borrower's continuation utility under the proposed mortgage contract with the initial expected utility for the borrower given by $a_0=A+C_0^L(r_0)-B_0$, which establishes (33).

Under the proposed mortgage contract and the borrower's optimal strategy $(C,\hat{Y},S)=(I^*,Y,0)$, the lender's expected utility equals

$$
E\left[\int\limits_0^{\tau(I^*,Y,0)} e^{-R_t}(dY_t-dI_t^*)+e^{-R_{\tau(I^*,Y,0)}\tau(I^*,Y,0)}L\,|\mathcal{F}_0\right],
$$

where

$$
\tau(I^*,Y,0)=\inf\left\{t:B_t=C_t^L\right\}=\inf\left\{t:\tilde{a}_t=A\right\}=\tau^*(Y),
$$

as the borrower's continuation utility, $\tilde{a}$, evolve according to the equation (87), e.g. as in the optimal allocation. Therefore, we conclude that the proposed mortgage contract implements the optimal allocation. $\square$

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